Toward an Axiomatic Characterization of the Smash Sum

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Abstract

We study the smash sum under the divisible sandpile model in $\mathbb{R}^d$, in which each point distributes mass equally to its neighbors. The smash sum as defined by an obstacle problem is conjectured to be uniquely characterized by a set of eight axioms. We prove several results that follow from the axioms and present two outlines sketching approaches that we believe will show that the axioms uniquely characterize the smash sum. We prove that a quadrature identity for superharmonic functions uniquely characterizes the smash sum and conjecture that the identity can be proven by our axioms. Using the known relationship between smash sums and Hele-Shaw flow, we outline a method of proving that the axioms are a unique weak solution to the Hele-Shaw problem.

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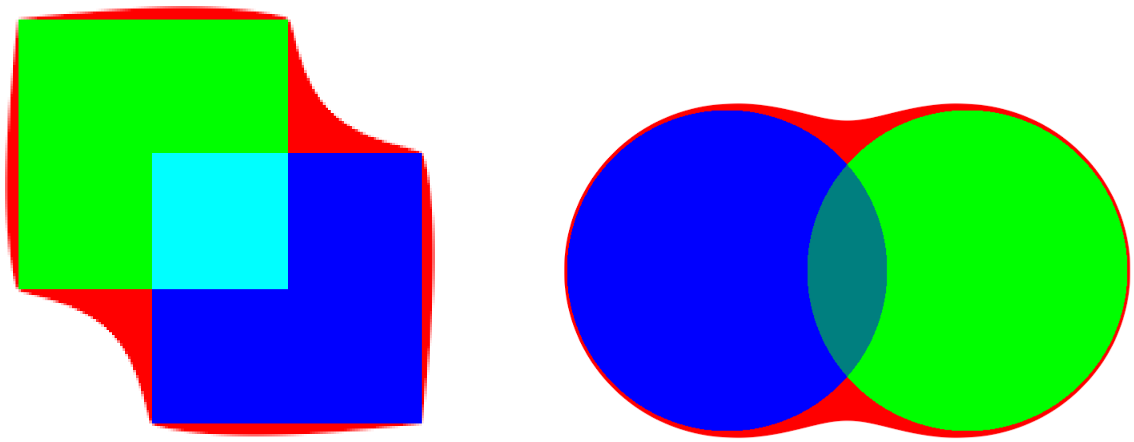

Figure 1.1: The smash sum of two squares with the overlap being a smaller square and the smash sum of two overlapping circles. The red is where the toppled mass ends up in the limit. Figures used with permission of Lionel Levine.

1 Introduction

The *smash sum*, $A \oplus_s B$, has several definitions, and in this paper we will be focusing on the *divisible sandpile* model. Intuitively, the divisible sandpile model begins with a sandpile with finite mass with height greater than one on some area of the sandpile. We want to topple the sandpile so that the height at each point is less than or equal to one. It is natural to expect that the total mass is preserved, the initial sandpile is contained in the toppled sandpile, and that the excess mass for where the height is greater than one to be mostly near the boundary of the sandpile to which it is closest. Determining the final sandpile is complicated, being defined as a solution to an obstacle problem. In this paper, we conjecture that the set resulting from the toppling can be defined with a set of simple axioms. We will begin by defining the smash sum under the divisible sandpile model as is done in [4].

We will first define the smash sum on $\mathbb{Z}^d$ and then extend the definition to $\mathbb{R}^d$. The divisible sandpile model uses continuous amounts of mass at each lattice point and the total mass is finite. A point is *full* if it has mass at least one and a full point can *topple* by keeping mass 1 for itself and distributing the excess equally to each of its $2d$ neighbors. If we topple full points at each time step, then as time goes to infinity, each point has mass ≤ 1. This is proven in [4].

Given finite sets $A, B \in \mathbb{Z}^d$, with the mass at point x equal to $1_A(x) + 1_B(x)$, we would like to get the limiting set that results from the toppling, the smash sum. We begin by defining an *odometer function*

$$u(x) = \text{total mass toppled from } x$$

Since each neighboring point $y \sim x$ emits an equal amount of mass to each of its $2d$ neighbors, the total mass x gains from its neighbors is $\frac{1}{2d}\sum_{y\sim x} u(y)$. So we get that

$$\Delta u(x) = \nu(x) - \sigma(x) \tag{1.1}$$

where Δ is the discrete Laplacian defined by $\Delta u(x) = \frac{1}{2d}\sum_{y\sim x}[u(y) - u(x)]$, $\sigma(x)$ is the initial amount of mass at x, and $\nu(x)$ is the final amount of mass at x.

To get the limiting shape we need to construct a function on $\mathbb{Z}^d$ that has Laplacian equal to $\sigma - 1$. Denote this function γ. An example of such a function γ, as seen in [4], is

$$\gamma(x) = -|x|^2 - \sum_{y\in\mathbb{Z}^d} g_1(x,y)\sigma(y) \tag{1.2}$$

where, in dimension $d \geq 3$, the Green's function $g_1(x,y)$ is the expected number of times a simple random walk started at x visits y. In dimension $d = 2$, we replace Green's function with the recurrent potential kernel. $|x|$ is the Euclidean norm. Then $\Delta(u+\gamma) \leq 0$ on $\mathbb{Z}^d$ (i.e. $u+\gamma$ is superharmonic). In fact, if we have a superharmonic function $f \geq \gamma$ then $f-\gamma-u$ is superharmonic on the finite domain $D = \{x \in \mathbb{Z}^d : \nu(x) = 1\}$ of full points and non-negative on D^c. Thus, $f-\gamma-u$ is non-negative everywhere by the minimum principle. This proves the following lemma from [4].

Lemma 1.1. *Let σ be a non-negative function on $\mathbb{Z}^d$ with finite support. Then the odometer function for the divisible sandpile started with mass $\sigma(x)$ at each site x is given by*

$$u = s - \gamma$$

where $\gamma(x) = -|x|^2 - \sum_{y\in\mathbb{Z}^d} g_1(x,y)\sigma(y)$ *and*

$$s(x) = \inf\{f(x) : \Delta f \leq 0 \text{ on } \mathbb{Z}^d, f \geq \gamma\}$$

is the least superharmonic majorant of γ.

This lemma can be reformulated as a *least action principle*. The following reformulation is stated and proved in [4].

Lemma 1.2 (Least Action Principle)**.** *Let σ be a nonnegative function on $\mathbb{Z}^d$ with finite support, and let u be its divisible sandpile odometer function. If $u_1 : \mathbb{Z}^d \to \mathbb{R}_{\geq 0}$ satisfies*

$$\sigma + \Delta u_1 \leq 1$$

then $u_1 \geq u$.

This lemma gives a method for extending the problem to $\mathbb{R}^d$. Suppose σ is a function of $\mathbb{R}^d$ which gives the initial mass at each point. By analogy to (1.2), the *obstacle* is defined to be

$$\gamma(x) = -|x|^2 - \int_{\mathbb{R}^d} g(x,y)\sigma(y)dy \tag{1.3}$$

where $g(x,y)$ is Green's function on $\mathbb{R}^d$ proportional to $-\log|x-y|$ in dimension $d=2$ and to $|x-y|^{2-d}$ in dimension $d \geq 3$. We set

$$s(x) = \inf\{f(x) : f \text{ continuous, superharmonic and } f \geq \gamma\}.$$

A function f is *superharmonic* in $\mathbb{R}^d$ if it satisfies the mean value property that $f(x) \geq (1/v)\int_{\mathcal{B}(x,v)} f dy$ where $\mathcal{B}(x,v)$ is an open ball of volume v centered at x. It is a well known result that if $f \in C^2$ then the mean value property implies $\Delta f \leq 0$. The *odometer function* for σ is $u = s - \gamma$ and the final domain is

$$D = \{x \in \mathbb{R}^d : s(x) > \gamma(x)\} \tag{1.4}$$

This set D is the *noncoincidence set for the obstacle problem* with obstacle γ.

Definition 1.3. If A, B are bounded open sets in $\mathbb{R}^d$ with boundaries of volume 0, then we define the *smash sum* of A and B as

$$A \oplus_s B = A \cup B \cup D \tag{1.5}$$

where D is given by (1.4) and $\sigma = 1_A + 1_B$.

Throughout the paper we will be referring to the smash sum defined in Definition 1.3 as the actual smash sum or the smash sum defined by the obstacle problem.

If h is a superharmonic function on $\mathbb{Z}^d$, and σ represents the mass configuration for the divisible sandpile then $\sum_{x\in\mathbb{Z}^d} h(x)\sigma(x)$ only decreases when there is a toppling. Therefore,

$$\sum_{x\in\mathbb{Z}^d} h(x)\nu(x) \leq \sum_{x\in\mathbb{Z}^d} h(x)\sigma(x)$$

where ν is the final mass configuration.

We would like an analogous inequality for the domain D. To do so we require the following proposition from [4].

Proposition 1.4. *Let σ be a C^1 function on $\mathbb{R}^d$ with compact support, such that $\mathcal{L}(\sigma^{-1}(1)) = 0$. Let D be given by (1.4). Then for any function $h \in C^1(\bar{D})$ which is superharmonic on D,*

$$\int_D h(x)dx \leq \int_D h(x)\sigma(x)dx$$

Proposition 1.5. *If σ has compact support, $\sigma \geq 1$ on supp(u), and continuous almost everywhere then*

$$\int_D h(x)dx \leq \int_D h(x)\sigma(x)dx$$

for all integrable superharmonic functions h on D, which is given by (1.4).

Proof. Fix $\epsilon > 0$. σ is continuous almost everywhere so there exist C^1 functions σ_0, σ_1 such that $\sigma_0 \leq \sigma \leq \sigma_1$ with $\int_{\mathbb{R}^d}(\sigma_1 - \sigma_0)dx < \epsilon$. We can approximate σ by $\hat{\sigma} = (1/2)(\sigma_0 + \sigma_1)$ Scaling by a factor of $1 + \delta$ for $\delta < \epsilon$, we can make $\mathcal{L}(\hat{\sigma}^{-1}(1)) = 0$, so $\hat{\sigma}$ satisfies the hypotheses of Proposition 1.4. Now applying Proposition 1.4, we get

$$\int_D h(x)dx \leq \int_D h(x)(1+\delta)\hat{\sigma}(x)dx,$$

for all integrable superharmonic functions h on D. As $\epsilon \to 0$, $\hat{\sigma} \to \sigma$ and $\delta \to 0$ giving us

$$\int_D h(x)dx \leq \int_D h(x)\sigma(x)dx \tag{1.6}$$

□

Equation (1.6) is a *quadrature inequality* for $D \in \mathbb{R}^d$. The smash sum given by Definition 1.3 satisfies this inequality because $\sigma = 1_A + 1_B$.

2 Definitions and Axioms

2.1 Definitions

Definition 2.1. The *volume* of a set A, denoted vol(A), is

$$\text{vol}(A) = \mathcal{L}(A),$$

where $\mathcal{L}$ is the d-dimensional Lebesgue measure.

Definition 2.2. A and B are equivalent, denoted $\cong$, if

$$\text{vol}(A\Delta B) = \text{vol}((A \cup B) - (A \cap B)) = 0.$$

That is, their symmetric difference has volume zero.

Definition 2.3. The *saturation* of an open set A with respect to the volume measure is defined to be

$$\text{sat}(A) = \bigcup_{\mathcal{B}(x,v)\in\mathcal{G}} \mathcal{B}(x,v)$$

where $\mathcal{G} = \{\mathcal{B}(x,v) : \text{vol}(\mathcal{B}(x,v) - A) = 0\}$ with $\mathcal{B}(x,v)$ being open balls with center $x \in \mathbb{R}^d$ and volume $v \in \mathbb{R}$.

This means that to saturate A, we add all open balls $\mathcal{B}(x,v)$ such that $\text{vol}(\mathcal{B}(x,v) - A) = 0$. The saturation of A contains A because A is open, so for all $x \in A$ there exists $\mathcal{B}(x,v) \subsetneq A$. This results in a set that contains A, has the same volume as A, and cannot be enlarged holding these two properties. We can choose a representative of the equivalence classes to be the saturation with respect to the volume measure.

The definition of saturation immediately gives us the following proposition.

Proposition 2.4. *A is saturated if and only if there exists B such that $A = sat(B)$ or $sat(A) = A$.*

Proposition 2.5. $A^\circ \subset sat(A) \subset \bar{A}$

Proof. Let $x \in A^\circ$. A is open, so there exists an open ball $\mathcal{B}(x,v) \subset A$. Clearly $\text{vol}(\mathcal{B}(x,v) - A) = 0$, so the $x \in \text{sat}(A)$. To show the other inclusion suppose $z \in \mathcal{B}(y,v)$ such that $z \in \bar{A}^c$. Because $\mathcal{B}(y,v)$ is an open set, there exists $\mathcal{B}(z,v') \in \mathcal{B}(y,v)$ such that $\mathcal{B}(z,v') \in \bar{A}^c$. This implies that $\text{vol}(\mathcal{B}(y,v) - A) > 0$. Therefore, any open ball that we add to A to saturate it must be contained in the closure, $\bar{A}$. Thus, $A^\circ \subset \text{sat}(A) \subset \bar{A}$.

□

Corollary 2.6. *If $vol(\partial A) = 0$ then $vol(A) = vol(sat(A))$.*

An alternative way of thinking about the saturation of an open set A is the largest set open set that contains A and has the same volume.

Proposition 2.7. *If $vol(\partial A) = 0$ then $sat(A) = \mathcal{U}$ where*

$$\mathcal{U} = \bigcup_{U \supset A, U\, open, vol(U)=vol(A)} U.$$

Proof. The union given above is clearly open. Let $x \in U$ for some $U \in \mathcal{U}$ then there exists $\mathcal{B}(x,v) \subset U$. Since $A \subset U$ and has the same volume, $\text{vol}(\mathcal{B}(x,v)-U) = \text{vol}(\mathcal{B}(x,v)-A) = 0$. Therefore, x is in a ball used to saturate A. $\square$

Proposition 2.8. *Suppose $vol(\partial A), vol(\partial B) = 0$. $A \cong B$ if and only if $sat(A) = sat(B)$.*

Proof. ($\Rightarrow$) $A \cong B$ gives us that $\text{vol}(A) = \text{vol}(A \cup B)$. So we have that $A \subset \text{sat}(A) \subset \text{sat}(A \cup B)$. By Proposition 2.7, we have that the saturation of a set is the largest open set that contains A and has the same volume, so $\text{sat}(A) = \text{sat}(A \cup B)$. Similarly, we can show that $\text{sat}(B) = \text{sat}(A \cup B)$. Therefore, $\text{sat}(A) = \text{sat}(B)$.

($\Leftarrow$) $\text{sat}(A)\Delta A \subset \partial A$. Since $\text{vol}(\partial A) = 0$, it follows that $A \cong \text{sat}(A)$. Similarly, we can show that $B \cong \text{sat}(B)$. Since $\cong$ is an equivalence relation, $A \cong B$. $\square$

2.2 Axioms

Let $\mathcal{A} = \{\text{sat}(B) : \text{vol}(B) < \infty, \text{vol}(\partial B) = 0, B \text{ open}\}$ be the family of saturated bounded, open sets in $\mathbb{R}^d$ with boundaries of volume zero. Let $\oplus$ be a function such that

$$\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}.$$

We want to characterize the actual smash $\oplus_s : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ as defined in the introduction with a set of axioms, which is the main conjecture of the paper.

Conjecture 2.9. *If $\oplus : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ satisfies the following axioms for $A, B, C \in \mathcal{A}$*

(1) $A \cup B \subseteq A \oplus B$

(2) $A \oplus B = B \oplus A$

(3) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

(4) $vol(A \oplus B) = vol(A) + vol(B)$

(5) If $p \in \mathbb{R}^d$ then $(A + p) \oplus (B + p) = (A \oplus B) + p$

(6) If M is a $d \times d$ orthogonal matrix, then $MA \oplus MB = M(A \oplus B)$

(7) If A is connected and $B \subset A$ then $A \oplus B$ is connected.

(8) If $A_n \uparrow A$, that is $A_n \subseteq A_{n+1}$ and $\cup_{n \geq 1} A_n = A$, then $A_n \oplus B \uparrow A \oplus B$

then $A \oplus B = sat(A \oplus_s B)$, where $\oplus_s$ is the smash sum defined by Definition 1.3.

Throughout the paper we will refer to the axioms that $\oplus$ satisfies as Axioms (1) - (8).

Remark 2.10. We believe that Axiom (7) follows from the others and could be omitted. We were not able to prove this, so it was left as an axiom.

3 Direct Consequences of the Axioms

Proposition 3.1. *If A and B are disjoint sets (i.e. $A \cap B = \emptyset$) then $A \oplus B = A \cup B$.*

Proof. $\mathrm{vol}(A \cup B) = \mathrm{vol}(A) + \mathrm{vol}(B)$ since A and B are disjoint. Since $A \cup B \subseteq A \oplus B$ are open sets with the same volume by axioms (1) and (4), it follows by properties of open sets that $A \cup B$ agrees with $A \oplus B$ up to measure 0. □

Proposition 3.2. *If $B \subseteq C$ then $A \oplus B \subseteq A \oplus C$*

Proof. Let $C_1 = C - \bar{B}$, then $A \oplus (B \cup C_1) \cong A \oplus C$. By construction B and C_1 are disjoint so $(A \oplus B) \oplus C_1 \cong A \oplus C$. Then by Axiom (1), we have that $A \oplus B \subset A \oplus C$. □

Proposition 3.3. *Suppose $\oplus$ satisfies axioms (1)-(8). If $\mathcal{B}(0, v)$ is an open ball centered at 0 with volume v, then*

$$\mathcal{B}(0, v) \oplus \mathcal{B}(0, v') = \mathcal{B}(0, v + v').$$

Proof. If M is a $d \times d$ orthogonal matrix then by axiom (6) and the fact that a ball centered at the origin is rotational invariant,

$$\mathcal{B}(0, v) \oplus \mathcal{B}(0, v') = M\mathcal{B}(0, v) \oplus M\mathcal{B}(0, v') = M(\mathcal{B}(0, v) \oplus \mathcal{B}(0, v')).$$

So we have that $\mathcal{B}(0, v) \oplus \mathcal{B}(0, v')$ is rotationally invariant. The only sets that are rotationally invariant about the origin and are connected (to satisfy Axiom (7)) are balls, implying that

$$\mathcal{B}(0, v) \oplus \mathcal{B}(0, v') = \mathcal{B}(0, v + v')$$

in order to satisfy Axiom (4). □

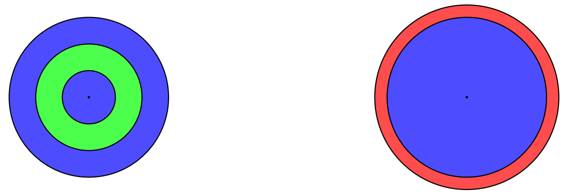

Figure 3.1: We are smashing a ball and annulus as in Proposition 3.5. The figure on the left has mass 1 in the blue area and mass 2 in the green area. After the toppling occurs, we get the ball on the right, where the red indicates how much the ball expanded.

Using Axiom (5) we get the following corollary.

Corollary 3.4. $\mathcal{B}(x, v) \oplus \mathcal{B}(x, v') = \mathcal{B}(x, v + v')$

Proposition 3.5. *If* $v_1 \geq v_2 \geq v_3$ *then* $\mathcal{B}(0, v_1) \oplus [\mathcal{B}(0, v_2) - \overline{\mathcal{B}(0, v_3)}] = \mathcal{B}(0, v_1 + v_2 - v_3)$.

Proof. If M is a $d \times d$ orthogonal matrix then by axiom (6) and the fact that a ball and an annulus centered at the origin are rotationally invariant,

$$\begin{aligned}\mathcal{B}(0, v) \oplus [\mathcal{B}(0, v_2) - \overline{\mathcal{B}(0, v_3)}] &= M\mathcal{B}(0, v) \oplus M[\mathcal{B}(0, v_2) - \overline{\mathcal{B}(0, v_3)}] \\ &= M(\mathcal{B}(0, v) \oplus [\mathcal{B}(0, v_2) - \overline{\mathcal{B}(0, v_3)}])\end{aligned}$$

So $\mathcal{B}(0, v) \oplus [\mathcal{B}(0, v_2) - \overline{\mathcal{B}(0, v_3)}]$ is rotationally invariant. The only sets that are rotationally invariant about the origin and are connected are balls, implying that

$$\mathcal{B}(0, v_1) \oplus [\mathcal{B}(0, v_2) - \overline{\mathcal{B}(0, v_3)}] = \mathcal{B}(0, v_1 + v_2 - v_3)$$

□

Corollary 3.6. *If* $v_1 \geq v_2 \geq v_3$ *then* $\mathcal{B}(x, v_1) \oplus [\mathcal{B}(x, v_2) - \mathcal{B}(x, v_3)] = \mathcal{B}(x, v_1 + v_2 - v_3)$.

Proposition 3.7. *If* $A_n \subset A$ *and vol*$(A_n) \to$ *vol*(A), *then* $\cup_n (A_n \oplus B) \cong A \oplus B$

Proof. By Proposition 3.2, $A_n \oplus B \subset A \oplus B$. By Axiom (4), volume is preserved so we also have that $\text{vol}(\cup_n (A_n \oplus B)) = \text{vol}(A \oplus B)$. This implies that $\cup_n (A_n \oplus B) \cong A \oplus B$. □

Remark 3.8. Proposition 3.7 is a continuity result that is similar to Axiom (8). For all the results in this section, we do not need to use Axiom (8). It might be possible to drop the axiom, but we believe that it will be necessary in the Hele-Shaw flow approach to proving Conjecture 2.9, so we leave it as an axiom.

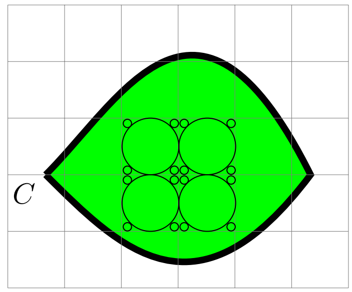

Figure 3.2: To prove Theorem 3.10 we tile C, which is in green, with cubes. We then approximate the cubes in C with disjoint balls and use associativity. As the volume of the cubes gets smaller and the number of balls approximating the cubes increases, this approximation approaches C.

Using Proposition 3.7, we get the following corollary.

Corollary 3.9. *If $A_n \subset A$ and vol$(A_n) \to$ vol(A), then sat$(\cup_n(A_n \oplus B)) =$ sat$(A \oplus B)$*

We now prove that the operation defined by the axioms is unique.

Theorem 3.10. *Fix $\epsilon > 0$. If $\oplus$ and $\boxplus$ both satisfy axioms (1)-(8) and $A \oplus B = A \boxplus B$ when $A \in \mathcal{A}$ and B is an open ball with vol$(B) < \epsilon$, then $A \oplus C = A \boxplus C$ for all $A, C \in \mathcal{A}$.*

Proof. Let $I_\delta = [0, \delta]$ and let

$$\mathcal{Q} = \{Q : Q = (a_1 + I_\delta) \times \cdots \times (a_d + I_\delta) \text{ s.t. } a_i \in \mathbb{Z}\}$$

be a family of cubes in $\mathbb{R}^d$ of volume δ^d that partition $\mathbb{R}^d$. For any δ, there exists $n(\delta) < \infty$ of cubes are entirely contained in C. Let

$$P_1(\delta), \ldots, P_{n(\delta)}(\delta) \subsetneq C$$

be the union of the cubes with volume δ^d, in $\mathcal{Q}$, and entirely contained in C. As $\delta \to 0$, $\text{vol}\left(\cup_{i=1}^{n(\delta)} P_i(\delta)\right) \to \text{vol}(C)$.

Let m be a decreasing function such that $m(\delta) \to \infty$ as $\delta \to 0$. For any δ, let $U_i^\delta = \cup_{j=1}^{m(\delta)} B_i^j \subsetneq P_i(\delta)$ be the union of $m(\delta)$ disjoint open balls such that $\text{vol}\left(U_i^\delta\right) = \text{vol}(P_i(\delta))$ as $\delta \to 0$. If $\delta < \epsilon^{1/d}$, then B_i^j has volume less than ϵ.

So we have

$$\bigcup_{i=1}^{n(\delta)}\left(U_i^\delta\right)\subset\bigcup_{i=1}^{n(\delta)}P_i(\delta)\subset C$$

Let $C_\delta=\bigcup_{i=1}^{n(\delta)}\left(U_i^\delta\right)$ so by Proposition 3.1 and associativity (Axiom (3)),

$$\begin{aligned}
A\oplus C_\delta &= A\oplus\left(U_1^\delta\oplus\cdots\oplus U_{n(\delta)}^\delta\right)\\
&= A\oplus\left(B_1^1\oplus\cdots B_1^{m(\delta)}\oplus\cdots\oplus B_{n(\delta)}^1\oplus\cdots\oplus B_{n(\delta)}^{m(\delta)}\right)\\
&= \left(A\oplus B_1^1\right)\oplus\left(B_1^2\oplus\cdots B_1^{m(\delta)}\oplus\cdots\oplus B_{n(\delta)}^1\oplus\cdots\oplus B_{n(\delta)}^{m(\delta)}\right)\\
&= \left(A\boxplus B_1^1\right)\oplus\left(B_1^2\oplus\cdots B_1^{m(\delta)}\oplus\cdots\oplus B_{n(\delta)}^1\oplus\cdots\oplus B_{n(\delta)}^{m(\delta)}\right)\\
&\vdots\\
&= A\boxplus B_1^1(\delta)\boxplus\cdots B_1^{m(\delta)}(\delta)\boxplus\cdots\boxplus B_{n(\delta)}^1\boxplus\cdots\boxplus B_{n(\delta)}^{m(\delta)}\\
&= A\boxplus\left(U_1^\delta\boxplus\cdots\boxplus U_{n(\delta)}^\delta\right)\\
&= A\boxplus C_\delta
\end{aligned}$$

So when $\delta<\epsilon^{1/d}$, $A\oplus C_\delta=A\boxplus C_\delta$. Since $\delta\to 0$ implies $n(\delta)\to\infty$ and $m(\delta)\to\infty$, $\mathrm{vol}(U_i^\delta)\to\mathrm{vol}(P_i(\delta))$ and $\mathrm{vol}(\bigcup_{i=1}^{n(\delta)}P_i(\delta))\to C$. Therefore, $\mathrm{vol}(C_\delta)\to\mathrm{vol}(C)$. Applying Proposition 3.7 we get $A\boxplus C_\delta\to A\boxplus C$ and $A\oplus C_\delta\to A\oplus C$, implying that $A\oplus C=A\boxplus C$ for all A,C. □

This theorem gives us that it is sufficient to prove that the axioms characterize the smash sum for $A,B\in\mathcal{A}$ for $\mathrm{vol}(B)<\epsilon$, greatly simplifying the amount of work that needs to be done.

4 Quadrature Approach

4.1 Quadrature Inequality Characterizes the Smash Sum

Theorem 4.1. *Let $A,B,D\in\mathcal{A}$ such that $A,B\subseteq D$ and*

$$\int_D h(y)dy\le\int_A h(y)dy+\int_B h(y)dy \tag{4.1}$$

for all superharmonic, integrable on D functions where dy is the differential with respect to the d-dimensional Lebesgue measure. Then we have that $D\cong A\oplus_s B$.

Proof. Taking $h = 1$, we get that the $\text{vol}(D) \leq \text{vol}(A) + \text{vol}(B)$ and when $h = -1$ that $\text{vol}(D) \geq \text{vol}(A) + \text{vol}(B)$, so

$$\text{vol}(D) = \text{vol}(A) + \text{vol}(B)$$

We want any two domains D that satisfy (4.1) to be equivalent up to measure zero. To do this we define the function

$$u(x) = \int_A g(x-y)dy + \int_B g(x-y)dy - \int_D g(x-y)dy$$

where g is Green's function in $\mathbb{R}^d$ proportional to $\log|x-y|$ in dimension $d = 2$ and to $|x-y|^{2-d}$ in dimension $d \geq 3$. $u(x)$ is well defined because A, B, D are bounded. Using the fact that Green's function satisfies $\Delta \int g(x-y)f(y)dy = -f(x)$, we get

$$\Delta u = -1_A - 1_B + 1_D.$$

If we take $h(x') = g(x - x')$ for a fixed x, we get that

$$\Delta h(x') = \Delta g(x - x') = -\delta_0(x - x') \leq 0$$

So by the quadrature inequality (4.1), we have the $u(x) \geq 0$.

The least action principle, Lemma 1.2, gives us that the odometer function w

$$w(x) = \inf\{v(x) : \Delta v \leq -1_A(x) - 1_B(x) + 1,\ v \geq 0\}$$

is a solution to the obstacle problem with obstacle γ such that $\Delta\gamma = 1_A + 1_B - 1$. Clearly, $u(x) \in \{v(x) : \Delta v \leq -1_A(x) - 1_B(x) + 1,\ v \geq 0\}$ we have $w(x) \leq u(x)$.

When $x \notin D$ then we have that $h(y) = g(x - y)$ is harmonic on D. By the quadrature property, $\int_D h(y)dy = \int_A h(y)dy + \int_B h(y)dy$, implying that $u(x) = 0$ when $x \notin D$. So we have $\text{supp}(u) \subseteq \bar{D}$.

Let $C = \{x \in \mathbb{R}^d : w(x) > 0\} \subseteq \{x \in \mathbb{R}^d : u(x) > 0\}$ so $C \subseteq \text{supp}(u)$. C is the noncoincidence set for the obstacle problem with obstacle γ. Since A, B, C, D are open, $A, B, C \subseteq D$ and $\text{vol}(D) = \text{vol}(A) + \text{vol}(B) = \text{vol}(A \oplus_s B)$, it follows that

$$A \cup B \cup C = A \oplus_s B \cong D.$$

$\square$

Theorem 4.1 gives us that the quadrature identity characterizes the smash sum. So a path to proving that the axioms uniquely characterize the smash sum could be showing that the axioms give the quadrature identity.

4.2 Outline to proving the Quadrature Identity from the Axioms

One path to showing that the axioms characterize the smash sum is to prove that the axioms give us the quadrature inequality (4.1) for A arbitrary and B small. Then we can appeal to Theorem 4.1 and Theorem 3.10 (which shows that the axioms define the smash sum for A, C arbitrary if it exists for A arbitrary and B small) and prove Conjecture 2.9.

Lemma 4.2. *Suppose $B, C \subset A$, $u \geq 0$ and supp$(u) \subset A$. If $1_C = 1_B + \Delta u$ then*

$$\int_C h\ dx \leq \int_B h\ dx$$

for $h \in C^2(A)$, integrable on A, and $\Delta h \leq 0$.

Proof.

$$\begin{aligned}\int_C h(x)\ dx &= \int_A h(x) 1_C(x)\ dx \\ &= \int_A h(x)(1_B(x) + \Delta u)\ dx \\ &= \int_B h(x)\ dx + \int_A [\Delta h(x)] u(x)\ dx\end{aligned}$$

The last equality comes from Green's formula, $\int_A h\Delta u\ dx - \int_A \Delta h u\ dx = \int_{\partial A} h\frac{\partial u}{\partial n}\ dS - \int_{\partial A}\frac{\partial h}{\partial n} u\ dS$, because the boundary terms vanish. We have $u \geq 0$ and $\Delta h \leq 0$, so $\int_A [\Delta h(x)] u(x)\ dx \leq 0$, giving us $\int_C h\ dx \leq \int_B h\ dx$. $\square$

Conjecture 4.3. *If $C = (A \oplus B) - A$ then there exists $u \geq 0$ such that*

$$1_C = 1_B + \Delta u.$$

Conjecture 4.4. *Suppose $B, C \subset A$, $u \geq 0$ and supp$(u) \subset A$. If $1_C = 1_B + \Delta u$ then $A \oplus B = A \oplus C$.*

If we have can prove Conjectures 4.3 and 4.4 then Theorem 4.1 would follow. Suppose that $C = (A \oplus B) - A$. Then we have that $A \cup C = A \oplus B$ so

$$\int_{A\oplus B} h(x)dx = \int_A h(x)dx + \int_C h(x)dx.$$

By Conjecture 4.3 there exists $u \geq 0$ such that $\mathrm{supp}(u) \subset A \oplus B$ such that $1_C = 1_B + \Delta u$. This allows us to use Lemma 4.2, which gives us that

$$\int_{A\oplus B} h(x)dx = \int_A h(x)dx + \int_C h(x)dx \leq \int_A h(x)dx + \int_B h(x)dx.$$

5 Hele-Shaw Flow Approach

5.1 Hele-Shaw Flow

Hele-Shaw flow can be defined in $\mathbb{R}^d$ but we will describe it in $\mathbb{R}^3$ to give intuition for the case that has physical meaning. Suppose we have two parallel flat planes parallel to the plane $x_3 = 0$ in $\mathbb{R}^3$ separated by distance h on the x_3 axis. Hele-Shaw flow is the flow of the fluid we inject into the gap at point z over time t. The case that is most relevant to this problem is when there is an open set Ω_0 of fluid between the planes, representing a cylinder $\Omega_0 \times [0, h]$. If we inject fluid of volume ϵ into the gap at point $z \in \Omega_0$ then we have that the Hele-Shaw flow results in $\Omega_0 \oplus_s \mathcal{B}(z, \epsilon)$. In fact, the actual smash sum is known to behave exactly like Hele-Shaw flow for any open sets Ω_0 and z. [4]

One approach to proving Conjecture 2.9 is to show that the axioms imply that the sets behave as Hele-Shaw flow and then appeal to Theorem 3.10 to get that the axioms uniquely characterize the actual smash sum. After stating the problem more precisely, we will present a rough outline that we believe would show this is the case.

We now state the Hele-Shaw flow problem more formally. We begin by stating the following *Hele-Shaw equation*

$$\tilde{\mathbf{V}} = -\frac{h^2}{12\mu} \bigtriangledown p. \tag{5.1}$$

where $\tilde{\mathbf{V}} = \left(\frac{1}{h}\int_0^h V_1 dx_3, \frac{1}{h}\int_0^h V_2 dx_3\right)$, the integral mean of the reference velocity $\mathbf{V} = (V_1, V_2)$, h is the distance between the two planes, μ is the fluid viscosity, and p is pressure. Here $\tilde{\mathbf{V}}$ and p depend only on x_1 and x_2. The derivation of this equation from the Navier-Stokes equation is omitted in this paper (see [2] for details).

Let $\Omega(t)$ be the bounded, open, simply connected set in the (x_1, x_2)-plane occupied by the fluid at time t. We consider injection through a single well that is placed at the origin. We assume that the source of the fluid is of constant strength Q. The dimensionless

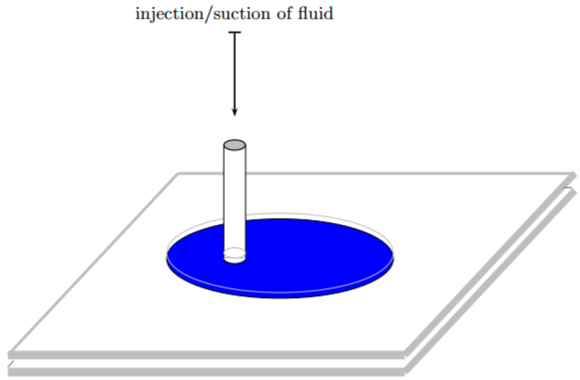

Figure 5.1: Hele-Shaw flow when the blue circle is the initial set, Ω_0, of liquid between the parallel planes and we are injecting fluid at some point inside Ω_0. The figure is from [2].

pressure p is scaled in order to have 0 correspond to the atmospheric pressure. We set $\Gamma(t) \equiv \partial\Omega(t)$. Before any fluid is injected, at time $t = 0$, the domain is $\Omega(0) = \Omega_0$. The potential function p is harmonic in $\Omega(t) - \{0\}$ and

$$\Delta p = Q\delta_0(z), \quad z = x_1 + ix_2 \in \Omega(t) \tag{5.2}$$

where $\delta_0(z)$ is the Dirac distribution with support at the origin. The zero surface tension dynamic boundary condition is

$$p(z,t) = 0 \text{ when } z \in \Gamma(t) \tag{5.3}$$

The resulting motion of the free boundary $\Gamma(t)$ caused by the fluid injection is given by the fluid velocity $\mathbf{V}$ on $\Gamma(t)$. The normal velocity in the outward direction is

$$v_n = \mathbf{V}|_{\Gamma(t)} \cdot \mathbf{n}(t)$$

where $\mathbf{n}(t)$ is the unit outer normal vector to $\Gamma(t)$. We then rewrite this law of motion in terms of the potential function and using (5.1) after a suitable rescaling we get the *kinematic* boundary condition

$$\frac{\partial p}{\partial \mathbf{n}} = -v_n, \tag{5.4}$$

where $\frac{\partial p}{\partial \mathbf{n}} = \mathbf{n} \cdot \bigtriangledown p$ is the outward normal derivative of p on $\Gamma(t)$.

Equations (5.2) - (5.4) are the *Hele-Shaw problem.*

5.2 Uniqueness of Weak Solutions of Hele-Shaw Flow

Definition 5.1. A solution to the Hele-Shaw problem, $\{\Omega(t) : 0 \leq t < \infty\}$, is said to be *weak* if there exists a function $u(z,t)$ such that

$$u \geq 0$$

$$u(z,t) = 0 \quad \text{for } z \notin \Omega(t)$$

$$1_{\Omega(t)} = 1_{\Omega(0)} + t\delta_0 + \Delta u$$

where δ_0 is Dirac's distribution supported at the origin.

$u(z,t)$ is related to the p and $\mathbf{V}$ in the strong solution by the *Baiocchi transform*, replacing p in (5.2) with $u(z,t) = \int_0^t p(z,\tau)\, d\tau$. The relationship between u and $\mathbf{V}$ then comes from (5.4).

Theorem 5.2. *Given any bounded open set Ω_0 there exists a unique weak solution $\{\Omega(t) : 0 \leq t < \infty\}$ with $\Omega(0) = \Omega_0$ (uniqueness in the strict sense that $\Omega(t)$ are required to be saturated.*

This theorem is proven in [2].

Conjecture 5.3. *$\Omega(t) = A \oplus \mathcal{B}(z,t)$ is a weak solution to the Hele-Shaw problem.*

The idea is that we can prove from Axioms (1) - (8) the existence of a function u that satisfies Definition 5.1. Then by Theorem 5.2 and Theorem 3.10 we have that the Axioms uniquely characterize the actual smash sum.

5.3 Outline using Hele-Shaw Flow

We now outline an approach to proving Conjecture 5.3 that was proposed by Lionel Levine. Let C_0^∞ be the set of smooth functions on $\mathbb{R}^d$ with compact support and let $A, B \in \mathcal{A}$.

Lemma 5.4. *If $\int_A \phi\, dx = \int_B \phi\, dx$ for all ϕ in C_0^∞, then $A \cong B$.*

Proof. Let D be a ball contained in $B - \bar{A}$. Let ϕ be a smooth positive bump function supported on $\bar{D}$. Then

$$0 = \int_A \phi\, dx < \int_B \phi\, dx.$$

This contradicts the hypothesis that $\int_A \phi\, dx = \int_B \phi\, dx$, so there is no such ball D. Therefore, $B - \bar{A}$ is empty. Similarly, $A - \bar{B}$ is empty. $\mathrm{vol}(\partial A) = \mathrm{vol}(\partial B) = 0$ because $A, B \in \mathcal{A}$ so $A \cong B$. $\square$

There is another characterization of Hele-Shaw flow with injection at z (See chapter 3 of [2] for details): Suppose Ω_t for $t \geq 0$ is a family of domains such that

$$\frac{d}{dt} \int_{\Omega_t} \phi\, dx = h_t(z) \tag{5.5}$$

where h_t is the harmonic extension of ϕ inside Ω_t. Given Ω_0 and solving the differential equation (5.5) gives $\int_{\Omega_t} \phi\, dx = \int_{\Omega_0} \phi\, dx + \int_0^t h_s(z)\, ds$. So by Lemma 5.4, Ω_t is determined up to measure zero.

The goal is to show from Axioms (1) - (8) that (5.5) holds when $\Omega_t = A \oplus B_t$, where $B_t = \mathcal{B}(z,t)$ is the ball of volume t centered at z. We want to show that (5.5) holds when z is outside of Ω_t. Then we need to prove that that the left side of (5.5) is a harmonic function of z inside Ω_t. To do this we want to establish an approximate mean value property for the function

$$f_t^A(z) = \int_{A \oplus B(z,t)} \phi - \int_A \phi.$$

We believe that the continuity axiom, Axiom (8), should give that in the limit as t goes to 0 we can replace $B(z,t)$ by a cube $Q(z,t)$ of volume t centered at z for $z \in A$: that is, $(1/t) \left(\int_{A \oplus B(z,t)} \phi - \int_{A \oplus Q(z,t)} \phi \right)$ tends to 0 as t tends to 0.

Now we work on two scales: tiny (t) and small (s) with $t < s$. Suppose that $t = s^2$. We can write B_s as a smash sum in 2 ways:

(1) $B_s = B_t \oplus ... \oplus B_t$ (with s/t summands)

(2) $B_s = C_1 \oplus ... \oplus C_m \oplus D$ where the C_i are disjoint cubes of volume t, and $D = B_s -$ (union of C_i) has small volume (going to 0 as t goes to 0). The number of summands m is approximately s/t.

By (1) and associativity, $f_s^A(z_0) = f_t^A(z_0) + f_t^{A \oplus B_t}(z_0) + f_t^{A \oplus B_t \oplus B_t}(z_0) + ...$ (with s/t summands)

The idea is to use the continuity axiom to show that the right side is close to $(s/t) f_t^A(x_0)$. When we attempted to do this, we got that the difference was of order s instead of order t, which we need.

When considering (2) and associativity,

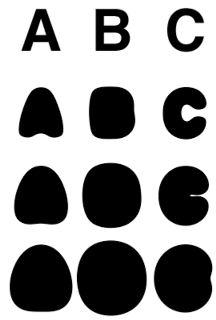

Figure 6.1: An example of Conjecture 6.3. We are smashing A, B, C with themselves. The second row shows the smash sum with 2 summands, the third row with 3 summands, and the fourth row with 4 summands. Figure used with permission from Lionel Levine.

$$f_s^A(z_0) = f_t^A(z_1) + f_t^{A+C_1}(z_2) + \ldots(\text{with } m \text{ summands}) + \int_D \phi.$$

Using the continuity axiom, we want to approximate the right side by changing all superscripts to A and disregard the last term because D has small volume. Then it should become a Riemann sum approximating the integral $(1/t)\left(\int_{B_s} f_t^A(z)\right)$.

Then we should have an approximate equality $f_t^A(z_0) = (1/s)\left(\int_{B_s} f_t^A(z)\right)$, which is the mean value property we were looking for. The goal is to quantify the error and then take s and t to 0 to show that the derivative in (5.5) exists and satisfies an exact mean value property (hence is a harmonic function of z).

6 Conjectures

Conjecture 6.1. *If A, B are bounded open sets, $k \in \mathbb{R}^+$, and $\oplus$ satisfies the axioms then*

$$kA \oplus kB = k(A \oplus B)$$

Conjecture 6.2. *Let A be an arbitrary bounded open set and $C_\epsilon \subsetneq A$ be small open set with volume ϵ. If $\mathcal{B}(x, \epsilon)$ is an open ball with volume ϵ centered at x, the center of mass of C_ϵ then*

$$vol([A \oplus C_\epsilon]\Delta(A \oplus \mathcal{B}(x, \epsilon)]) = o(\epsilon)$$

We want to use Conjecture 6.2 in section 5.3 when we replace balls with cubes.

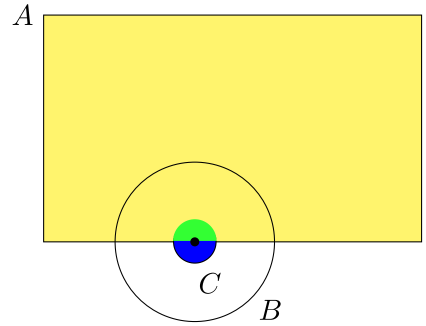

Figure 6.2: If A, B, C are as in Conjecture 6.5, where C is a ball centered on the boundary of A and B is a bigger ball centered on the boundary of A. The centers of B and C are denoted by the black dot. Before the toppling takes place, the yellow and blue areas have mass 1 and the green area has mass 2. After we take the smash sum of A and C, we conjecture that most of the mass in $(A \oplus C) - A$ is in B.

Conjecture 6.3. *Let C_ϵ be an open set with volume ϵ and center of mass x, then smashing C_ϵ to itself $n-1$ times (i.e. there are n summands) then*

$$vol[(C_\epsilon \oplus \cdots \oplus C_\epsilon) \, \Delta \, \mathcal{B}(x, n\epsilon)] = o(n\epsilon)$$

Conjecture 6.4. *Let A and $C \subsetneq A$ be arbitrary open sets. Then for all $x \in \partial A$ and for all $\delta > 0$, $\mathcal{Y} = \{y \in (A \oplus C) - A : |x - y| < \delta\}$ such that $vol(\mathcal{Y}) > 0$.*

That is, the mass does not spread out from only one part of A. We believe that we need this result to prove the following conjecture.

Conjecture 6.5. *If A is an arbitrary open set and B is an open ball with center $x \in \partial A$ and volume ϵ, and C a small open ball with the same center as B with volume $\delta < \epsilon$ satisfies*

$$\lim_{\delta \downarrow 0} \frac{vol([(A \oplus C) - A] \cap B)}{\delta} = 1.$$

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