

A Geometric Understanding of Ricci Curvature in
the Context of Pseudo-Riemannian Manifolds

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APRIL 2012

BACHELOR OF ARTS, CORNELL UNIVERSITY, 2012

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1 Introduction

The Ricci curvature tensor of an oriented Riemannian manifold M measures the extent to which the volume of a geodesic ball on the surface differs from the volume of a geodesic ball in Euclidean space. This statement is made precise via the formula [1]:

$$\omega_{1,\dots,n} = \left(1 - \frac{1}{6} \sum_{i,j=1}^n R_{ij}(m)x_i x_j + O(|x|^3) \right) \omega_{1,\dots,n}(m) \quad (1)$$

where ω is the metric volume n -form for the metric tensor g defined on M and Riemann normal coordinates are used. For the pseudo-Riemannian manifolds of general relativity, the Ricci curvature tensor is typically approached from a purely formulaic perspective by means of a trace of the Riemannian curvature tensor. While this approach yields correct physical results in the form of the Einstein equations, it does not lead to any meaningful geometric intuition. In this paper, I show that Eq. 1 generalizes to four-dimensional pseudo-Riemannian manifolds, quantitatively qualifying the statement that the Ricci curvature tensor describes the extent to which a pseudo-Riemannian manifold in general relativity differs from flat, Minkowski spacetime. I present images from the Schwarzschild geometry to support this result pictorially and to lend geometric intuition to the abstract notion of Ricci curvature for the pseudo-Riemannian manifolds of general relativity.

2 Preliminary Definitions

The following definitions are taken from [1] and [2] unless otherwise noted.

Definition 1. *A smooth manifold is a set M together with a specified C^∞ structure on M such that the topology induced by the C^∞ structure is Hausdorff and paracompact. M is called orientable if its tangent bundle TM is orientable. An orientation for TM is also called an orientation for M . A smooth manifold M together with an orientation for M is said to be an oriented manifold.*

I will henceforth refer to a smooth oriented manifold simply as a manifold.

Definition 2. *A scalar product or metric tensor on a real finite dimensional vector space V is a nondegenerate symmetric bilinear form*

$$g : V \times V \rightarrow \mathbb{R}.$$

Definition 3. *A Riemannian metric tensor g is a nondegenerate, symmetric, positive definite tensor field on a manifold M . The pair (M, g) is referred to as a Riemannian manifold.*

Definition 4. *The index of a symmetric bilinear form g on V is the dimension of the largest subspace $W \subset V$ such that the restriction $g|_W$ is negative definite. The index is denoted $\text{ind}(g)$.*

Definition 5. A pseudo-Riemannian metric tensor g is a nondegenerate, symmetric tensor field with constant index on M . The pair (M, g) is referred to as a pseudo-Riemannian manifold.

Note that for the case of a pseudo-Riemannian manifold in general relativity, $\text{ind}(g) = 1$.

I now define the covariant derivative, which may be defined in several different ways. In this section, I will give the most physically intuitive definition of the covariant derivative, in terms of parallel transport [3].

Definition 6. Given a curve $\alpha(\lambda)$ in M , the covariant derivative $\nabla_u T$ of a tensor field T is defined by

$$\nabla_u T|_{\alpha(0)} = \lim_{\epsilon \rightarrow 0} \frac{T(\alpha(\epsilon))_{\text{parallel-transported to } \alpha(0)} - T(\alpha(0))}{\epsilon}.$$

Definition 7. A geodesic of a (pseudo)-Riemannian manifold is a curve α such that $\nabla_{\dot{\alpha}}(\dot{\alpha})|_{\alpha(t)} = 0$ for all t . In other words, a geodesic is a curve that parallel-transport its own tangent vector.

At this point, I wish to define the exponential map Exp_m , which takes an element X of the tangent space $T_m M$ and returns a point in M by following a geodesic starting at m with initial tangent vector X . The Fundamental Theorem of (pseudo)-Riemannian geometry establishes the existence of the Levi-Cevita connection on every (pseudo)-Riemannian manifold [4], which allows me to give an equivalent definition for a geodesic as a solution to the equation

$$\frac{d^2 x^\alpha}{d\lambda^2} + \sum_{\beta, \gamma} \Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0 \quad (2)$$

for any affine parameter λ . By the existence and uniqueness of ordinary differential equations, given $m \in M$, $X \in T_m M$, there exists a unique maximal geodesic $\gamma_X(\lambda)$ such that $\gamma_X(0) = m$, $d\gamma_X(d/d\lambda)|_0 = X$ [5]. Thus, I may define the exponential map according to

$$\text{Exp}_m(X) = \gamma_X(1). \quad (3)$$

I further define the notion of a coordinate vector field.

Definition 8. X is said to be a coordinate vector field at m if there exist constants a^1, \dots, a^n such that in a neighborhood of m ,

$$X = \sum_{\alpha} a^\alpha \frac{\partial}{\partial x^\alpha}.$$

Definition 9. The Riemann curvature tensor field R of a (pseudo)-Riemannian manifold M is given in terms of the covariant derivative by

$$R(X, Y) = \nabla_{[X, Y]} - [\nabla_X, \nabla_Y].$$

for tangent vectors $X, Y \in TM$. In local coordinates, one may define

$$\langle X^\delta, R_{X_\alpha X_\beta} X_\gamma \rangle = R^\delta_{\alpha\beta\gamma}$$

to be a coordinate expression of the Riemann curvature tensor.

Definition 10. The Ricci curvature tensor field $R_{\alpha\beta}$ is given by

$$R_{\alpha\beta} = \sum_{\gamma} R^\gamma_{\alpha\gamma\beta}.$$

Definition 11. Suppose that $\dim(M) = n$. The metric volume form induced by the metric tensor g is the n -form ω such that ω_m is the metric volume form on $T_m M$ matching the orientation. If (U, x) is a positively oriented chart on M , then

$$\omega|_U = \sqrt{|\det g|} dx^1 \wedge \dots \wedge dx^n.$$

Furthermore, all covariant derivatives of ω vanish for a (pseudo)-Riemannian manifold.

Note that much of the formalism of Riemannian geometry carries over to the pseudo-Riemannian case. It comes as little surprise, therefore, that the expansion of Eq. 1 applies to pseudo-Riemannian manifolds, as I will show in the following section.

3 Taylor Expansion of the Metric Volume Form

The following proof closely parallels that of A. Gray in [1] in the Riemannian case. In fact, no result up to and including Lemma 2.5 in [1] uses positive-definiteness of the metric tensor, so I may take this lemma as a starting point even in the case of pseudo-Riemannian manifolds:

Lemma 12. Take $m \in M$ and let $X_\alpha \in TM$ be coordinate vector fields that are orthonormal at m . Then,

$$(\nabla_{\alpha\beta}^2 X_\gamma)(m) + (\nabla_{\beta\alpha}^2 X_\gamma)(m) = -\frac{1}{3}(R_{X_\alpha X_\gamma} X_\beta)(m) - \frac{1}{3}(R_{X_\beta X_\gamma} X_\alpha)(m).$$

The remainder of the proof is essentially the same, but I must take extra care in dealing with raised vs. lowered indices when dealing with pseudo-Riemannian manifolds.

Let (M, g) be an analytic pseudo-Riemannian manifold of signature $(n, 1)$ (i.e. $\text{ind}(g) = 1$, $\dim(M) = n + 1$). Let ω be the metric volume form, defined in a neighborhood of m . Let X_0, \dots, X_n denote coordinate vector fields that are orthonormal at m , and let (x^0, \dots, x^n) be the corresponding normal coordinate system (here x^0 should be thought of as the time coordinate). Define

$$\omega_{0,\dots,n} \equiv \omega(X_0, \dots, X_n).$$

Then there exists a power series expansion

$$\omega_{0,\dots,n} = \omega_{0,\dots,n}(m) + \sum_{\alpha=0}^n (X_\alpha \omega_{0,\dots,n})(m) x^\alpha + \frac{1}{2} \sum_{\alpha,\beta=0}^n (X_\alpha X_\beta \omega_{0,\dots,n})(m) x^\alpha x^\beta + O(|x|^3). \quad (4)$$

For the first-order term, I have

$$(X_\alpha \omega_{0,\dots,n})(m) = (\nabla_\alpha \omega)(X_0, \dots, X_n)(m) + \sum_{\beta=0}^n \omega(X_0, \dots, \nabla_\alpha X_\beta, \dots, X_n).$$

But, all covariant derivatives of ω vanish, so the first term in the sum vanishes. Furthermore, $\nabla_\alpha X_\beta = 0$, so that the first-order term drops out altogether. For the second-order term,

$$\begin{aligned} \frac{1}{2} \sum_{\alpha,\beta=0}^n (X_\alpha X_\beta \omega_{0,\dots,n})(m) x^\alpha x^\beta &= \frac{1}{2} \sum_{\alpha,\beta=0}^n (\nabla_\alpha \nabla_\beta \omega)(X_0, \dots, X_n)(m) x^\alpha x^\beta \\ &\quad + \sum_{\alpha,\beta,\gamma=0}^n (\nabla_\alpha \omega)(X_0, \dots, \nabla_\beta X_\gamma, \dots, X_n)(m) x^\alpha x^\beta \\ &\quad + \frac{1}{2} \sum_{\alpha,\beta,\gamma,\delta=0}^n \omega(X_0, \dots, \nabla_\alpha X_\gamma, \dots, \nabla_\beta X_\delta, \dots, X_n)(m) x^\alpha x^\beta. \end{aligned}$$

Again, the first two terms of this sum vanish because all covariant derivatives of ω vanish. Furthermore, all first covariant derivatives $\nabla_\alpha X_\gamma$ vanish because the vector fields X_γ are coordinate vector fields. Hence, only second derivatives contribute, so I am left with

$$\frac{1}{2} \sum_{\alpha,\beta=0}^n (X_\alpha X_\beta \omega_{0,\dots,n})(m) x^\alpha x^\beta = \frac{1}{2} \sum_{\alpha,\beta,\gamma=0}^n \omega(X_0, \dots, \nabla_{\alpha\beta}^2 X_\gamma, \dots, X_n)(m) x^\alpha x^\beta.$$

Here, I project onto the basis of coordinate vectors, yielding

$$= \frac{1}{2} \sum_{\alpha,\beta,\gamma,\nu=0}^n \langle X^\nu, \nabla_{\alpha\beta}^2 X_\gamma \rangle \omega(X_0, \dots, X_{\gamma-1}, X_\nu, X_{\gamma+1}, \dots, X_n)(m) x^\alpha x^\beta.$$

By definition of ω , this vanishes unless $\nu = \gamma$, so the sum collapses to

$$= \frac{1}{2} \sum_{\alpha,\beta,\gamma=0}^n \langle X^\gamma, \nabla_{\alpha\beta}^2 X_\gamma \rangle \omega_{0,\dots,n}(m) x^\alpha x^\beta.$$

By Lemma 12, I have

$$(\nabla_{\alpha\beta}^2 X_\gamma)(m) + (\nabla_{\beta\alpha}^2 X_\gamma)(m) = -\frac{1}{3} (R_{X_\alpha X_\gamma} X_\beta)(m) - \frac{1}{3} (R_{X_\beta X_\gamma} X_\alpha)(m).$$

This yields,

$$\begin{aligned}
\frac{1}{2} \sum_{\alpha,\beta=0}^n (X_\alpha X_\beta \omega_{0,\dots,n})(m) x^\alpha x^\beta &= \frac{1}{2} \sum_{\alpha,\beta,\gamma=0}^n \langle X^\gamma, -\frac{1}{3} R_{X_\alpha X_\gamma X_\beta} \rangle \omega_{0,\dots,n}(m) x^\alpha x^\beta \\
&= -\frac{1}{6} \sum_{\alpha,\beta,\gamma=0}^n R^\gamma_{\alpha\gamma\beta} \omega_{0,\dots,n}(m) x^\alpha x^\beta \\
&= -\frac{1}{6} \sum_{\alpha,\beta=0}^n R_{\alpha\beta} \omega_{0,\dots,n}(m) x^\alpha x^\beta.
\end{aligned}$$

So indeed, I find

$$\omega_{0,\dots,n} = \left(1 - \frac{1}{6} \sum_{\alpha,\beta=0}^n R_{\alpha\beta} x^\alpha x^\beta + O(|x|^3) \right) \omega_{0,\dots,n}(m). \quad (5)$$

4 Application to the Schwarzschild Geometry

The Schwarzschild metric, which describes the geometry outside a spherical star or black hole of mass M , is given by the line element,

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \left(1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (6)$$

in units where both the speed of light and the Newton gravitational constant are set to 1, $c = G = 1$. The Christoffel symbols for this geometry are given in Appendix A. By spherical symmetry, analysis may be constrained to the x - y plane by setting $\theta = \pi/2$. With this, the geodesic equation of Eq. 2 yields,

$$\begin{aligned}
\frac{d^2 t}{d\lambda^2} + \frac{2M}{r^2} \left(1 - \frac{2M}{r} \right)^{-1} \frac{dt}{d\lambda} \frac{dr}{d\lambda} &= 0 \\
\frac{d^2 r}{d\lambda^2} + \frac{M}{r^2} \left(1 - \frac{2M}{r} \right) \left(\frac{dt}{d\lambda} \right)^2 - \frac{M}{r^2} \left(1 - \frac{2M}{r} \right)^{-1} \left(\frac{dr}{d\lambda} \right)^2 - (r-2M) \left(\frac{d\phi}{d\lambda} \right)^2 &= 0 \\
\frac{d^2 \phi}{d\lambda^2} + \frac{2}{r} \frac{dr}{d\lambda} \frac{d\phi}{d\lambda} &= 0
\end{aligned} \quad (7)$$

Given any point $m = (t, r, \pi/2, \phi)$, with $r \neq 0, 2M$, these geodesic equations induce a well defined exponential map from the tangent space $T_m M$ to the manifold M . We may further examine the geodesic balls that result from taking the exponential map at m in all directions. In flat Minkowski space, these geodesic balls are actually spheres. The images in Figure 1 show how geodesic balls in the Schwarzschild geometry gradually deform from perfect spheres as their radii increase.

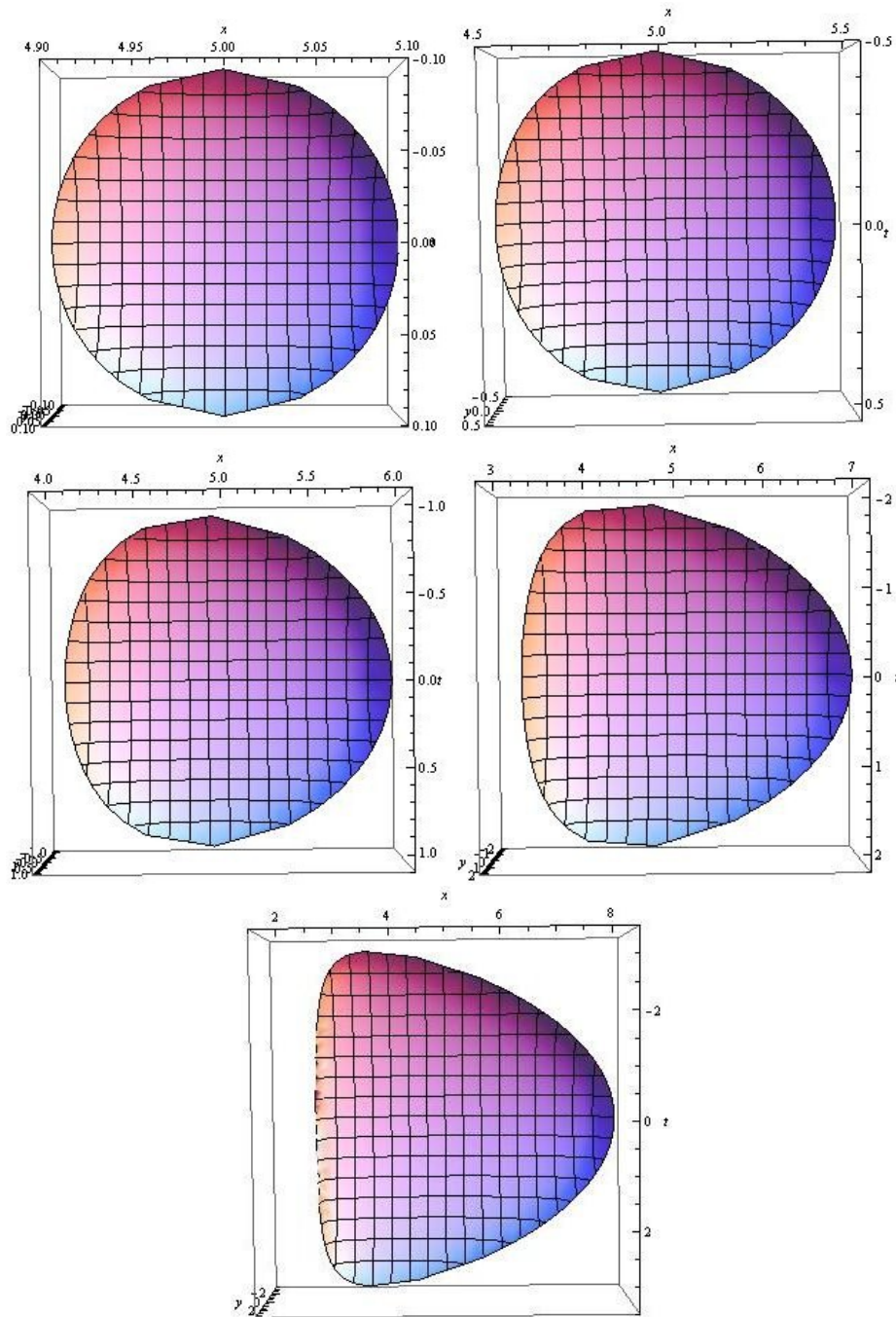


Figure 1: Geodesic balls outside $r = 2M$. Balls computed via the exponential map at the point $m = (0, 5M, \pi/2, 0)$ become more deformed as their radius increases ($\tau = 0.1, 0.5, 1, 2$ and $3M$, respectively).

As expected, the small geodesic ball of radius $0.1M$ is approximately spherical. As the radius increases to $3M$, the balls become progressively more misshapen. The images of Figure 2 tell a similar story, this time for geodesic balls inside the “Schwarzschild radius” of $r = 2M$.

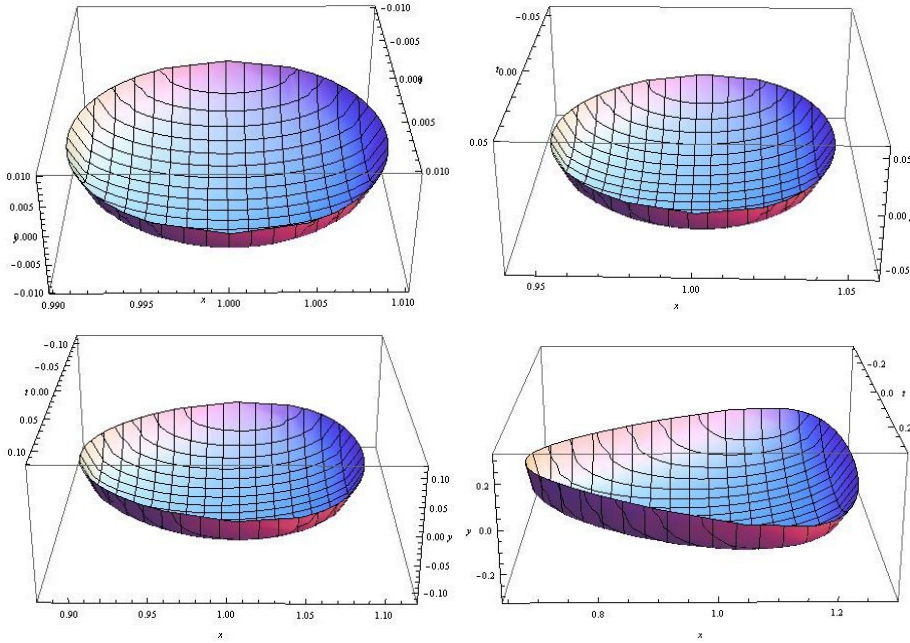


Figure 2: Geodesic balls inside $r = 2M$. Balls computed via the exponential map at the point $m = (0, M, \pi/2, 0)$ become more deformed as their radius increases ($\tau = 0.01, 0.05, 0.1$, and $0.3M$, respectively), just as in the case for $r > 2M$.

In both sets of images, the geodesics slow down as they approach the Schwarzschild radius, which may have been expected since $r \rightarrow 2M$ implies $t \rightarrow \infty$ in the Schwarzschild geometry. As $r \rightarrow 0, \infty$, the geodesic ball comes to more of a point. Given these images, it is not difficult to imagine that as the radius τ of a four-dimensional geodesic ball approaches 0 for any $r \neq 0, 2M$, the volume of the ball should approach $\frac{1}{2}\pi^2\tau^4 + O(\tau^7)$, in keeping with the results of §3 and the fact that $R_{\alpha\beta} = 0$ for the Schwarzschild geometry (c.f. Appendix A).

5 Conclusion

I have shown that the Taylor expansion of the metric volume form in terms of the Ricci curvature tensor applies to pseudo-Riemannian manifolds. This expansion provides an intuitive way to understand the Ricci curvature tensor of general relativity, quantifying the statement that the Ricci curvature measures

the extent to which a particular spacetime geometry differs from flat Minkowski space.

I have verified this result pictorially in the case of the Schwarzschild geometry, demonstrating that small geodesic balls are nearly perfect spheres but that they deform as the radius increases. Hence, the volume of such a geodesic ball differs from the volume of a geodesic ball in Minkowski space by an increasing amount as the radius of the ball increases.

6 Acknowledgements

I wish to express my profound gratitude to my adviser, Professor John Hubbard, for his guidance throughout all aspects of this project. I would also like to thank Professor Liam McAllister for our helpful discussions.

A Schwarzschild Geometry

The Schwarzschild metric is given by the line element in Eq. 6. This metric induces Christoffel symbols [6],

$$\begin{aligned}
 \Gamma_{tr}^t &= (M/r^2)(1 - 2M/r)^{-1}, & \Gamma_{r\theta}^\theta &= 1/r \\
 \Gamma_{tt}^r &= (M/r^2)(1 - 2M/r), & \Gamma_{\phi\phi}^\theta &= -\cos\theta \sin\theta \\
 \Gamma_{rr}^r &= -(M/r^2)(1 - 2M/r)^{-1}, & \Gamma_{r\phi}^\phi &= 1/r \\
 \Gamma_{\theta\theta}^r &= -(r - 2M), & \Gamma_{\theta\phi}^\phi &= \cot\theta \\
 \Gamma_{\phi\phi}^r &= -(r - 2M)\sin^2\theta.
 \end{aligned} \tag{8}$$

By Birkhoff's theorem, the Schwarzschild metric is the unique spherically symmetric solution to the vacuum Einstein field equations, $G_{\alpha\beta} = R_{\alpha\beta} = 0$ [3].

References

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