GRAPH COLORING February 9, 2013

Graph - a collection of vertices V and edges E (connection between a pair of vertices).

Vertex Coloring

Question: Given a map of some country divided into different territories, what is the fewest number of colors required to color each region in the map a different color so that no adjacent territories have the same color?

Let's try to color the map of the United States, divided up by states: see if you can find a coloring of it that uses the fewest possible colors. Can you color in this map with 3 colors so that no states sharing a border have the same color? How about 4? 5? Why or why not?

By trying to color near California, you will discover that 3 colors aren't enough. If you color cleverly, it is possible to use just 4 colors on this map. (And so of course, you can use 5!) We might next ask: 'so what about other maps? Might there be maps that need 5 colors? 6 colors? Is there any bound on how many colors are necessary?' When we try to answer this question we realize that it's going to be really hard to come up with *all possible maps*.

Encoding maps as graphs

- leave out details that don't matter to coloring
- examples: neither wiggling the border of Texas and Oklahoma so that it looks more rectangular, nor pushing in Florida's peninsula will change the problem. We can color these maps in exactly the same way.
- capture all of the important information: all states and which states share boundaries

Maybe you can guess how we will do this: we want a vertex for each state, and we'll draw an edge in whenever two states have a common boundary. Notice too that we only get planar graphs from maps like these. Translating our coloring problem into this new context we get the question: How many colors do we need to guarantee that for any planar graph we can color the vertices so that no two vertices joined by an edge are the same color? Is there an upperbound on the number of colors we need?

This might not seem to help our situation much, but it does! Still, this is a very difficult problem. In 1852 it was conjectured (guessed) by Francis Guthrie that you only ever need 4 colors. However, this was such a difficult problem that not until 1976 was anyone able to prove it! Appel and Haken's proof was the first famous mathematical proof that relied on a computer's help. There were too many cases to check by hand.

Exercise: Can you show that if you have a complete graph on n vertices (this means that every two vertices are connected by exactly one edge) that you need n colors?

Exercise: Can you show that the utility graph is 2-colorable?

Friends and Strangers- Edge Coloring We can use edge coloring to figure out the following classic problem: *How many people do you have to invite to a party to guarantee that there is at least a group of 3 friends or 3 mutual strangers?*

How can we model this with graphs?

Each person is a vertex, and between any two people there is an edge- we can say something about their relationship. If they are strangers, we can encode this by coloring the edge blue. If they are friends, by coloring it red.

Definition A graph is called *complete* when all pairs of vertices are joined by an edge. For example, the pentagram is complete. The complete graph with n vertices is called K_n.

Let's translate our Friends and Strangers problem into this language. What are we looking for?

What is the smallest n such that we can color the edges of K_n with 2 colors and guarantee that there is at least one blue triangle or red triangle?

Try for yourself! Is it possible to color K_4 with two colors and avoid a triangle all of the same color? How about K_5? K_6? K_7?

On Saturday, we showed that you need to invite just 6 people to your party to guarantee either three mutual strangers or three mutual friends. You can check out www.math.cornell.edu/$\sim$ mec/Winter2009/Thompson/cliques.html which has lovely pictures showing this.