FLOWS AND CONNECTIVITY April 13, 2013

Why do we care how connected a graph is?

- we don't want a network of computers (or websites) to fail if a few computers (or websites) get a virus or need maintenance.

Today we will examine:

Quantity of data going through a network vs. Cutting edges in a network

Want to get the following result:

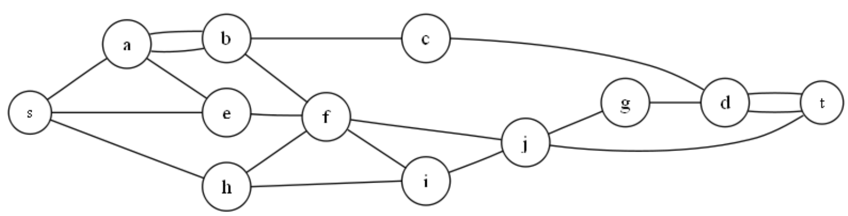
"Largest number of distinct routes between two special vertices in a graph"

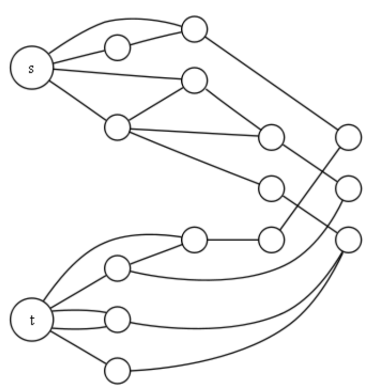
$$\|$$

"Minimal number of edges that can be cut to separate these two vertices"

Activity 1: For each of the graphs below:

- what is the minimal number of edges you have to cut to separate s from t?
- what is the maximum number of routes from s to t which do not share edges?





Flow problem

You can think of collection of pipes connecting an oil well and a refinery. We are given:

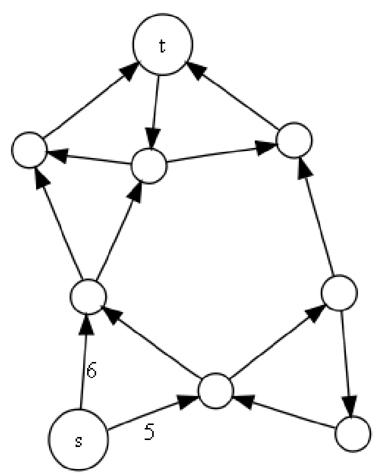
- a directed graph
- two special vertices: source, s, and sink, t.
- values on edges (called *capacity* = maximum allowed flow on that edge)

We want a flow, i.e. an assignment of non-negative numbers to each edge, such that:

- no bursting pipes: the number assigned to an edge should be less than its capacity
- pipes are not leaky: for each vertex, besides s and t, the total flow going in is equal to the total flow going out

Notice: The flow conservation means that the flow coming out of s is equal to the flow going into t, we will call this amount the *value of the flow*. In practice, we are often interested in the maximal flow.

Activity 2:
You are given the flow out of s. Fill in valid flow values for other edges given that the capacity of each edge is 6.



Is the flow coming out of s equal to the flow going into t?

Cutting edges

A *cut* is a separation of vertices of the graph into two components X and $V - X$, such that s is in X and t is in $V - X$. The *capacity of a cut* is the sum of the capacities of all the edges going from X to $V - X$.

Notice: Removing the edges between X and $V - X$ stops all flows from s to t. So,

"Maximal value of any flow" $\leq$ *"Minimal capacity of all cuts"*

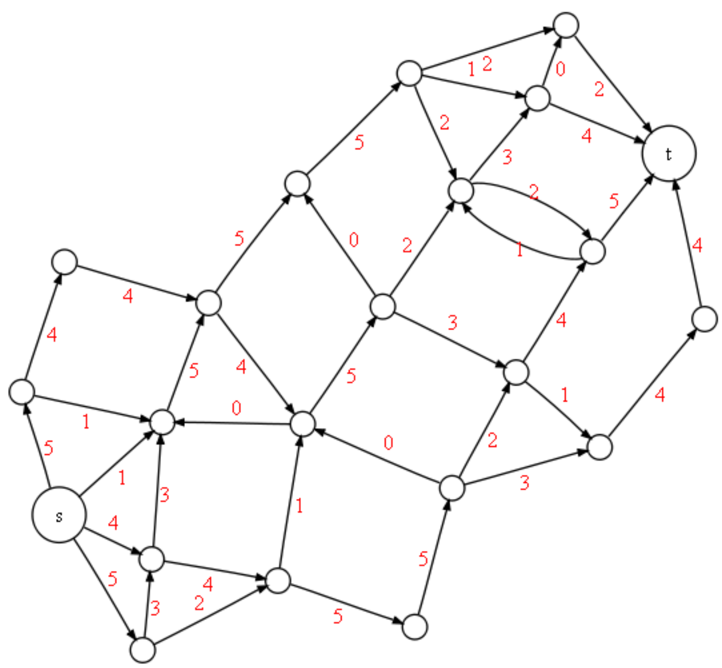
In fact, we have equality above!

Maximal flow $=$ *Minimal capacity of cuts*

Given a flow on a graph, we can build the set X corresponding to a cut of the same value recursively by:

- put s in X
- for every, v vertex in X and w a neighbor of v:
  if capacity$(v \to w) >$ flow$(v \to w)$, or
  if flow$(w \to v) > 0$,
  then add w to X.
- repeat step 2 until no more vertices can be added

Activity 3:
Assume all edges have capacity 5. Given the following flow on the graph, find the set X corresponding to this flow using the method described above.



Is the given flow maximal?