1 Questions

Question 1.1. Compare $(R1 \circ R2) \circ X_A$ and $R1 \circ (R2 \circ X_A)$.

Question 1.2. Which ones of the following operations are associative.

(i) Addition on the real numbers. Multiplication on the positive real numbers.

(ii) Division on the non-zero real numbers.

Question 1.3. Suppose an operation $*$ on X has two identity elements e and f. Use the identity property for both e and f to simplify $e * f$ into two different expressions, and conclude that $e = f$ necessarily.

Question 1.4. Let $X, *$ and e as in the previous definition but assume that $*$ is associative. Now let $x \in X$ with two inverses y and z. In other words, $x * y = y * x = e$ and $x * z = z * x = e$. Show that $z = z * (x * y) = (z * x) * y$ and conclude that $z = y$ and hence that an inverse under an associative operation is unique if it exists.

Question 1.5. Let $X, *$ and e as in the previous definition, but this time assume that X is finite, i.e. it only has finitely many elements, and that $*$ is associative. Now let $x \in X$ and $y \in X$ such that $x * y = e$. Consider the map $f : X \to X$ defined by $f(a) = y * a$ for $a \in X$.

(i) Show that f is **one-to-one**, a.k.a injective, by using an associativity trick on $x * f(a)$

(ii) Deduce that f is also onto because X is finite.

(iii) Deduce now that the previous statement implies that there exists an element $z \in X$ with $y * z = e$.

(iv) Show that $x = x * (y * z)$ implies that $x = z$ and $y * x = e$.

Question 1.6. Does the set of all symmetries of a square form a group?

Question 1.7. Does the set of all permutation of n elements form a group?

Question 1.8. Which one of these sets with operations are Groups?

1. The natural numbers with addition? With multiplication? How about substraction?

2. Same question with the rational numbers?

Forming a new group from old groups?

Question 1.9. Let $(\mathbb{Z}, +)$ be the additive integer group. Is $\mathbb{Z} \times \mathbb{Z} = \{(a, b) \mid a, b \in \mathbb{Z}\}$ a group? What would be its binary operation?

Question 1.10. Can you generalize this to any group $(G, *)$?

Question 1.11. Let M be the set of expressions of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c and d are real numbers. Define the binary operation $+$ on M by the following expression:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} + \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{pmatrix}.$$

Show that this operation makes M into a group. What is the identity in this case?

Question 1.12. Let O be the subset of M with elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad - bc \neq 0$, with operation $*$ defined:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} * \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}.$$

Show that this makes O into a group with identity. What is the identity in this case?

Question 1.13. What are the rotational symmetries of a 3-dimensional cube?