Knot Theory Worksheet 1 March 15th, 2014
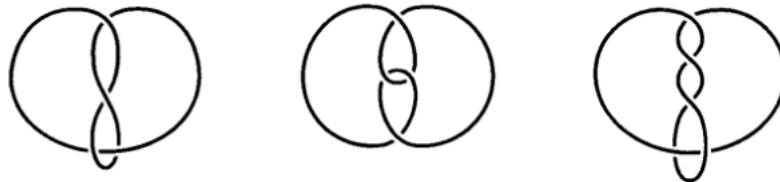
Knot Projections

1. Show that the following knot is the trefoil knot. Either construct it out of pipe cleaners and physically deform one to the other, or draw a sequence of pictures that show the deformation from one to the other.

2. Use pipe-cleaners to show that the following three projections are of the same knot (called the figure-eight knot).


3. Show that the following knot is actually the unknot (either through diagrams or pipe-cleaners).

4. Use your pipe-cleaners to create an unknot, trefoil, or figure-eight knot, draw an interesting projection of it, and pass it to a partner who will figure out which knot it is.

Knot Theory Worksheet 2 March 15th, 2014

Compositions of Knots and the Connect Sum

1. Here is a list of the prime knots which have projections with seven or fewer crossings:
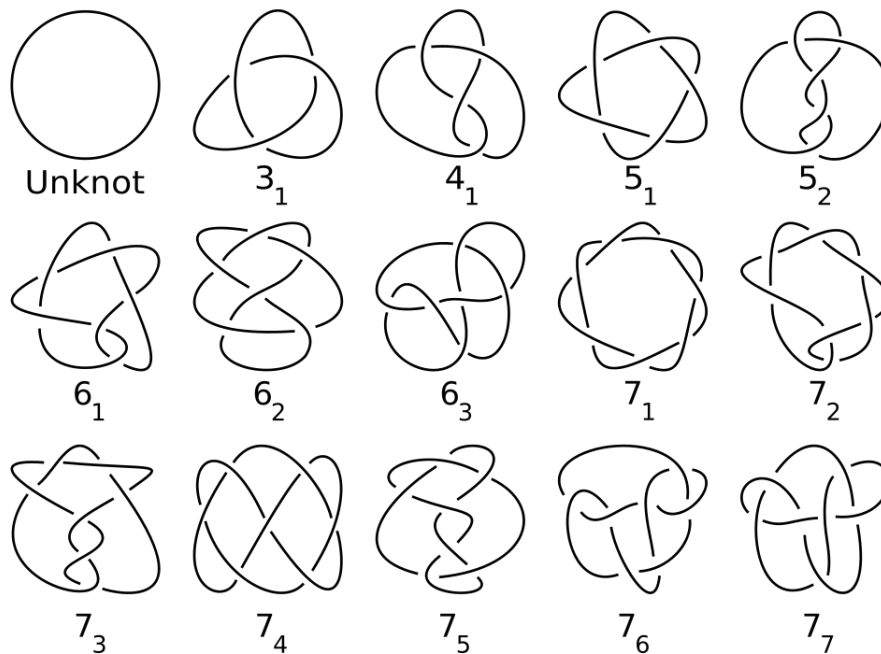
2. The following knots are composite knots. Determine which knots they can be decomposed as, using the chart above.
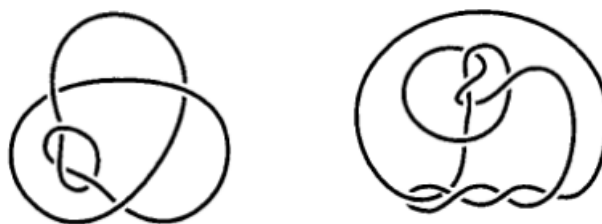

3. Create two of the small prime knots out of pipe cleaners, then connect sum them and give them to a partner to make sense of.

4. Here is an example of two distinct knots which are made of the same factor knots.

Knot Theory Worksheet 3 March 15th, 2014

Orientation and Composite Knots

1. Example: Knots (a) and (b) are connect sums where the orientations match, and knot (c) is a connect sum where the orientations do not match.
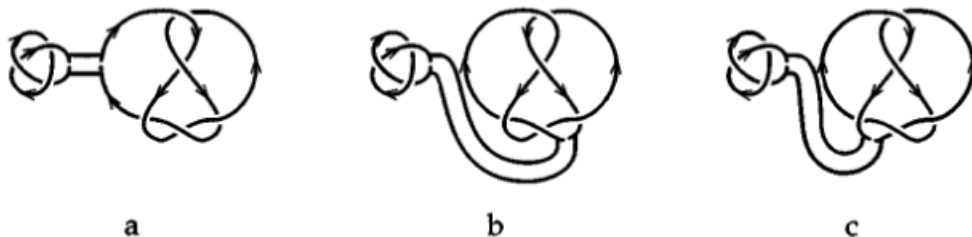
2. Example: Since both (a) and (b) are connect sums where the orientations match, they must represent the same knot. Here is a deformation showing why this is true:
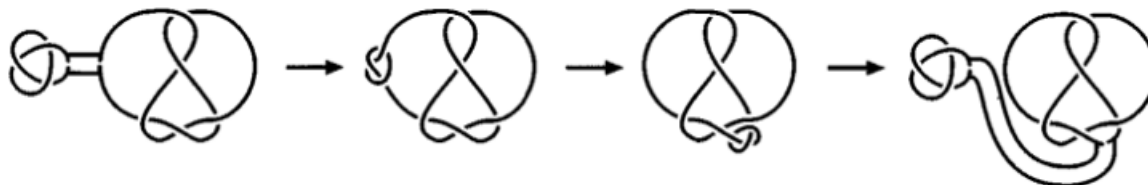

3. Show that the trefoil knot is *invertible*. That is, show that after choosing an orientation on the trefoil knot, it can be deformed into the opposite orientation.

4. Discuss with your group or partner why this implies that knots a,b, and c above are all equivalent.

5. Take a closer look at the two composite knots from before. Why is it possible for them to be different knots?

Knot Theory Worksheet 4   March 15th, 2014
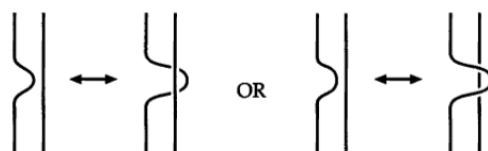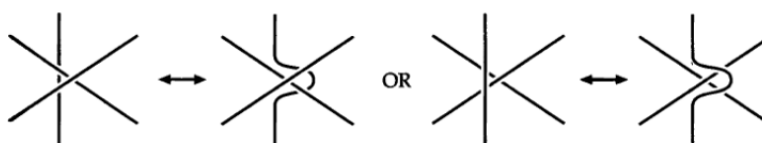
The Reidemeister Moves

1. Here are the three Reidemeister moves. We claim that these moves, together with planar isotopies (wiggling parts of the projection without changing the crossings) will allow us to move between any two projections of the same knot.
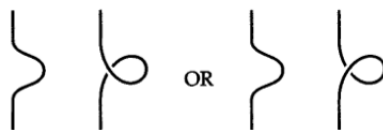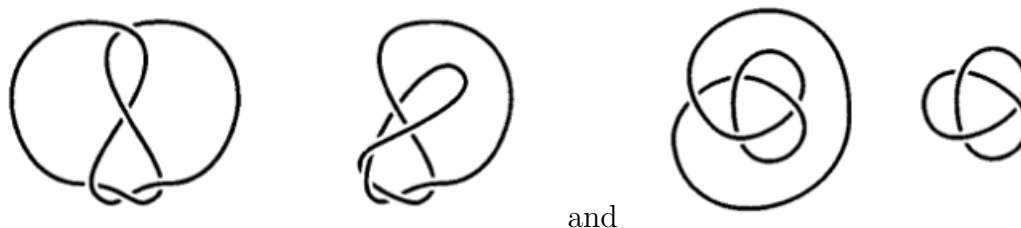

Type I Reidemeister move.

Type II Reidemeister move.

Type III Reidemeister move.

2. Sketch the sequence of Reidemeister moves and planar isotopies that shows that the first two projections are of the same knot, and the second two projections are of the same knot:

and

3. Use pipe cleaners to make the following knot, and show that it is the unknot. Consider the Reidemeister moves you use as you manipulate the knot. If you are feeling brave, write down the sequence of Reidemeister moves involved.

Knot Theory Worksheet 5 March 15th, 2014
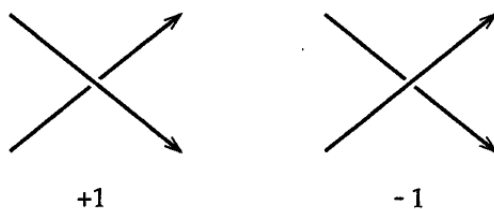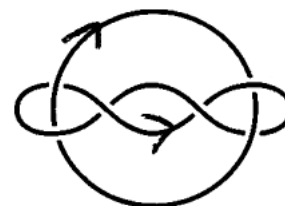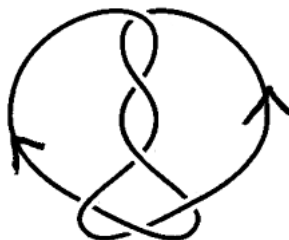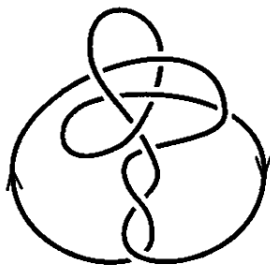
Links and Linking Number

1. Use pipe cleaners to show that the following link is splittable.

2. We use the following convention to compute the linking number of an oriented link:

   +1   − 1

3. Compute the linking number of the following links. Can you conclude anything from what you get?
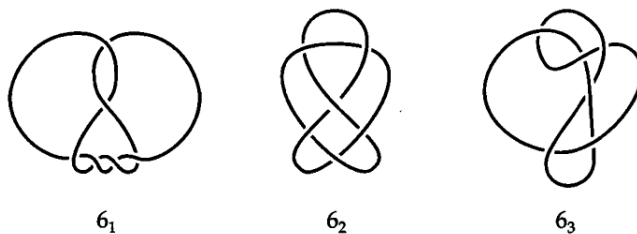
4. Use the Reidemeister moves to prove that linking number is an invariant of oriented links.

5. Construct the Borromean rings out of pipe cleaners. Check that removing any one of the three components makes the link splittable. Find a link with four components that has the same property. This generalization is called a Brunnian link.
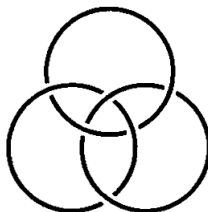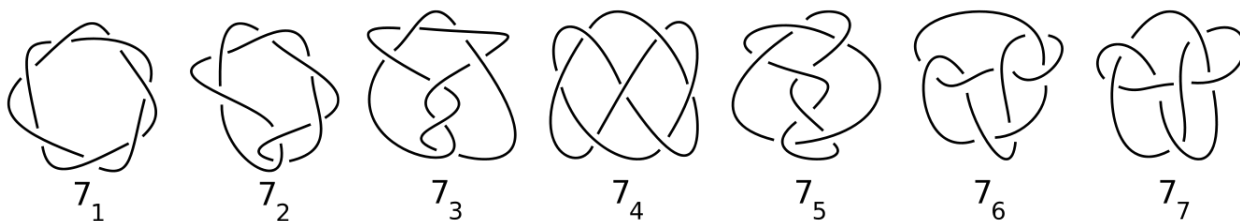
Knot Theory Worksheet 6 March 15th, 2014

Tricolorability

1. Determine which of the 6-crossing knots 6_1, 6_2, and 6_3 are tricolorable.

6_1 6_2 6_3

2. Use the Reidemeister moves to prove that tricolorability is an invariant of links.

3. Determine which of the 7-crossing knots are tricolorable.

7_1 7_2 7_3 7_4 7_5 7_6 7_7

4. Show that the composition of any knot with a tricolorable knot yields a new tricolorable knot.

5. Give an argument that shows that the figure-eight knot is not tricolorable. Conclude that the figure-eight knot and the trefoil knot are distinct knots. (Similarly, this proves that the figure-eight knot is not the unknot!!)
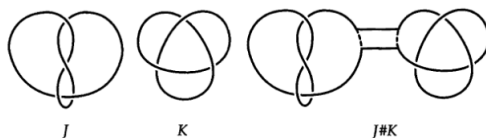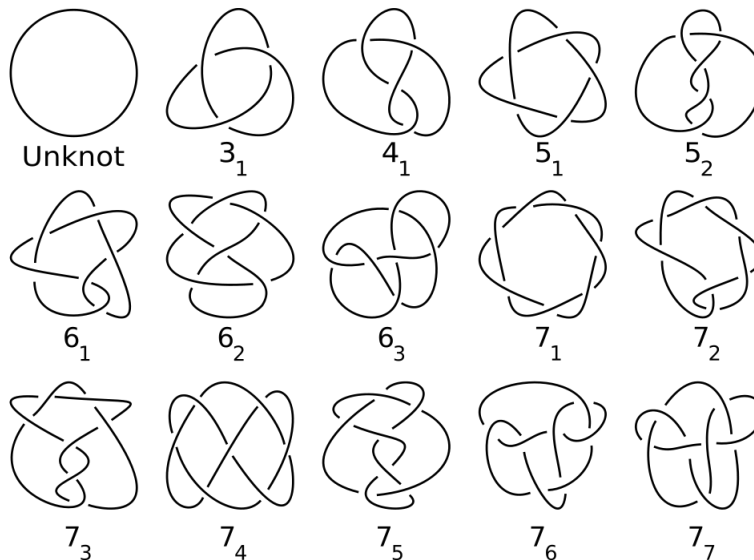
Knot Theory Review Worksheet

Some things to remember from last time

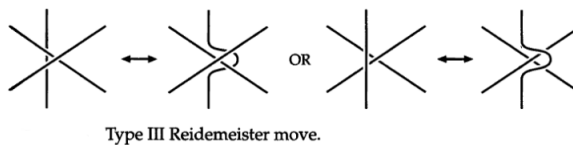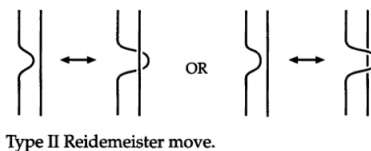1. The connect sum of two knots:

J   K   J#K

2. List of prime knots:

Unknot 3_1 4_1 5_1 5_2

6_1 6_2 6_3 7_1 7_2

7_3 7_4 7_5 7_6 7_7

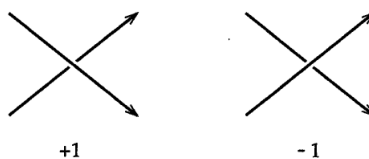3. Reidemeister moves:

OR

Type I Reidemeister move.

OR

Type II Reidemeister move.

OR

Type III Reidemeister move.
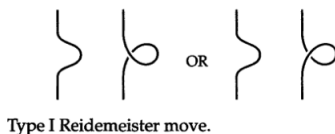
4. Linking Number: (Draw yourself a link with two components and calculate its linking number.)

+1   − 1

Knot Theory Worksheet 6

Tricolorability

1. Determine which of the 6-crossing knots 6_1, 6_2, and 6_3 are tricolorable.


6_1 6_2 6_3

2. Use the Reidemeister moves to prove that tricolorability is an invariant of links.

3. Determine which of the 7-crossing knots are tricolorable.

7_1 7_2 7_3 7_4 7_5 7_6 7_7

4. Show that the composition of any knot with a tricolorable knot yields a new tricolorable knot.

5. Give an argument that shows that the figure-eight knot is not tricolorable. Conclude that the figure-eight knot and the trefoil knot are distinct knots. (Similarly, this proves that the figure-eight knot is not the unknot!!)

Knot Theory Worksheet 7   March 22nd, 2014
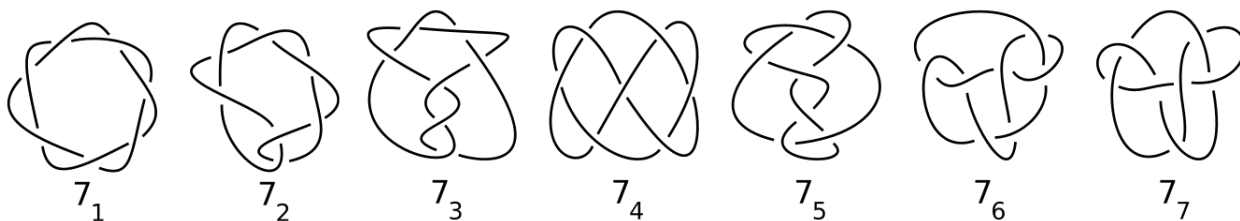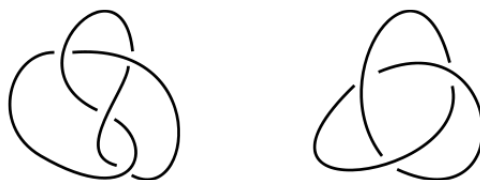
Unknotting Number

1. Definition: We say that a knot has **unknotting number** n if there exists a projection of the knot such that changing n crossings makes it into the unknot, and there is no projection with fewer changes that would have changed it into the unknot.

2. The knot 7_2 has unknotting number 1.
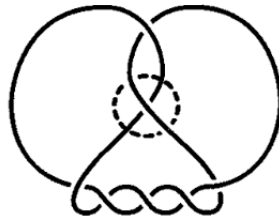
3. Compute the unknotting number of the figure-8 knot. Does this seem like something that should be easy to compute in general? Why or why not?

4. Does every knot have an unknotting number? Given a random projection of a knot, is there always a way to change some number of crossings and turn it into a projection of the unknot?

Knot Theory Worksheet 8 March 22nd, 2014

Unknotting Number

1. Here is the diagram that goes with the explanation of why every knot has a finite unknotting number. We begin by changing all of the crossings in the following way:

(a) Original projection. (b) Altered projection.

2. Then if we think of this knot in three dimensions, starting with a z-value of 1 and decreasing to a z-value of 0, we get the following diagrams, showing that we can change any projection into a projection of the unknot.
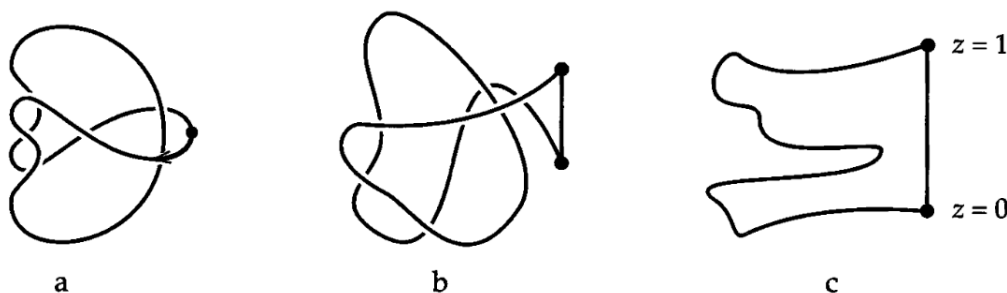

(a) Altered projection. (b) Partial side view. (c) Side view. The altered projection is the trivial knot.

3. Find an inequality that relates the unknotting number $u(K)$ and the minimum crossing number $c(K)$.

Knot Theory Worksheet 9 March 22nd, 2014
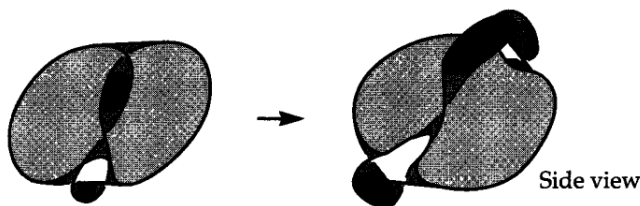
Seifert Surfaces

1. The goal is to create a 2-dimensional surface sitting in 3-dimensional space which has a given knot as its boundary. To construct this, we start with a projection of the desired knot, and choose an orientation on the projection. We then remove all of the crossings by connecting each of the strands coming into the crossing to the adjacent strand leaving the crossing.
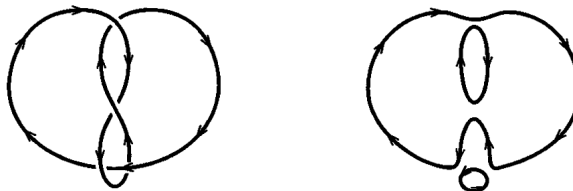
3. We are left with a set of disjoint circles. We imagine each circle as being the boundary of a 2-dimensional disk, but since we are allowed to work in 3-dimensions we place each of the disks at a different height so that they don't intersect one another.
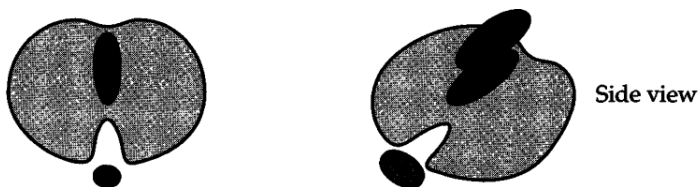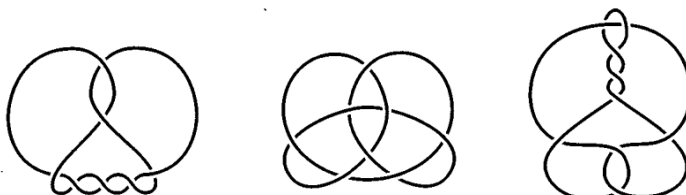
4. Finally, at each spot where there was originally a crossing we connect the two disks with a twisted strip.

5. As an example, the following surface has boundary the 6_3 knot.
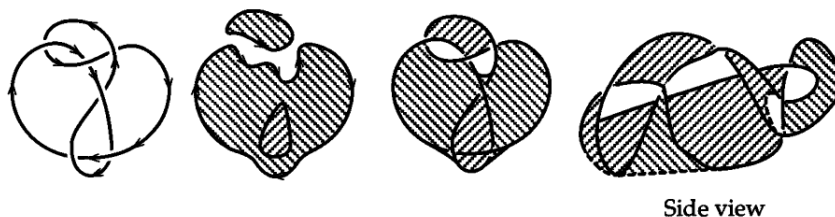
6. Use Seifert's algorithm to construct the Seifert surfaces which have the following knots as boundaries:

Knot Theory Worksheet 10   March 22nd, 2014

The Jones Polynomial

1. We are moving on to bigger and better invariants. What if we assigned each link a polynomial instead of a single number? We begin with the **bracket polynomial**. We would like it to satisfy a few rules:

   Rule 1: $<\bigcirc> = 1$

   Rule 2: $<\!\!\times\!\!> = A <)(> + B <\asymp>$

   $<\!\!\times\!\!> = A <\asymp> + B <)(>$

   **Rule 3:** $<L \cup \bigcirc> = C <L>$

2. But in particular, we need the bracket to be an invariant; unchanged under Reidemeister moves! In order to get the bracket to be invariant under Reidemeister II moves, we have to make the following choices for B and C:

   Rule 1: $<\bigcirc> = 1$

   Rule 2: $<\times> = A <)(> + A^{-1} <\asymp>$

   $<\times> = A <\asymp> + A^{-1} <)(>$

   Rule 3: $<L \cup \bigcirc> = (-A^2 - A^{-2})<L>$

3. Check that with these rules, the bracket polynomial is unchanged under a Reidemeister III move.

4. Compute the bracket polynomial of the Hopf link and the trefoil, both using the standard projections. (Do the Hopf link first, since you will use that result to compute the trefoil.)
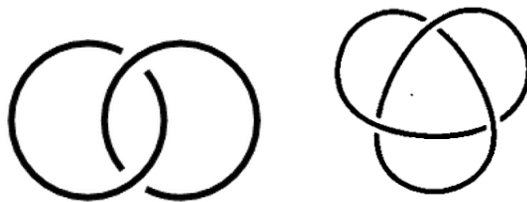
5. Unfortunately, the bracket polynomial is not a link invariant because it does not stay constant under Reidemeister I moves. Luckily, there is a fix...

# Knot Theory Worksheet 11 March 22nd, 2014

The Jones Polynomial

1. Let's first figure out how the bracket polynomial acts under Type I Reidemeister moves:

$$\langle \text{⌣} \rangle = A\langle \text{⌣} \rangle + A^{-1}\langle \text{⌣} \rangle$$
$$= A(-A^2-A^{-2})\langle \text{⌣} \rangle + A^{-1}\langle \text{⌣} \rangle$$
$$= -A^3\langle \text{—} \rangle$$

$$\langle \text{⌣} \rangle = A\langle \text{⌣} \rangle + A^{-1}\langle \text{⌣} \rangle$$
$$= A\langle \text{—} \rangle + A^{-1}(-A^2-A^{-2})\langle \text{—} \rangle$$
$$= -A^{-3}\langle \text{—} \rangle$$

2. So this tells us that each type I Reidemeister twist the projection of our knot will add a factor of $-A^{-3}$. If we could somehow figure out exactly how many such twists we had, we could multiply the bracket polynomial by $-A^3$ that many times, and then we would have an invariant!

3. Choose an orientation on your link projection. The **writhe** of the projection, $w(L)$ is the sum of a +1 or -1 for each crossing in the projection, where we make the following choices based on the orientation:

Positive crossing   Negative crossing
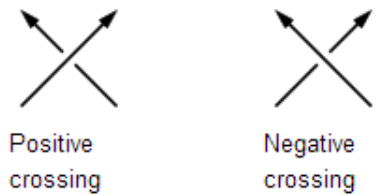
4. Calculate the writhe $w(L)$ of the following projection:
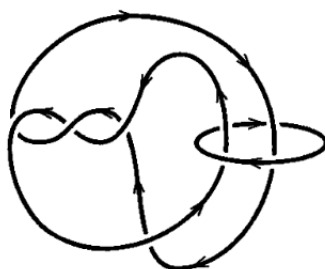
5. Show that the writhe is invariant under Type II and Type III Reidemeister moves and only changes by 1 under Type I moves.

6. The Jones Polynomial is an invariant of oriented links, given by $J(L) = (-A^3)^{-w(L)}\langle L \rangle$. Calculate the Jones Polynomial of the unknot, Hopf link, and trefoil.