Day 1: Markov Chains

1. Markov chain icebreaker: Have them stand is circle. First, I roll a six-sided die; hand the die to the person that many to the left and have them share their name and a fact about them. Then they roll the die and repeat. After a few rounds, start asking questions: Is it possible that Suzy will get the die 10 times before Joey gets it once? How many times, on average, do you think we will have to roll the die to get to everyone? Later, after they learn the definition of a Markov chain, ask them to write out why this icebreaker was a Markov chain, what the state space is, and what the transition probabilities are.

2. Introduce first example: Lecture. Cover the first page of the worksheet. Make sure everyone is on board with our first example, the frog and the lily pads. Then pass out Worksheet #1 and give them time to fill it out.

3. Review Worksheet #1: Quickly go over answers. Explain to them about random walks in n-dimensions. Conclude section: What are the ingredients for a Markov chain? State space, transition probabilities. What has to be true about the transition probabilities?

4. $P^n(x, y)$:
   - If a frog starts on lily pad A, what are all the paths it can take in two hops? What is the probability taking each of these paths? (Let them calculate.)
     - path $A \to A \to A$, prob $1/3 \cdot 1/3 = 1/9$
     - path $A \to A \to B$, prob $1/3 \cdot 1/3 = 1/9$
     - path $A \to A \to C$, prob $1/3 \cdot 1/3 = 1/9$
     - path $A \to B \to A$, prob $1/3 \cdot 1/2 = 1/6$
     - path $A \to B \to C$, prob $1/3 \cdot 1/2 = 1/6$
     - path $A \to C \to C$, prob $1/3 \cdot 1 = 1/3$
   - Add up all these numbers. What do you get?
   - Using what you just calculated, what's the probably of ending up on lily pad A after 2 hops? Lily pad B? Lily pad C?
     - Lily pad A: $1/9 + 1/6 = 5/18$
     - Lily pad B: $1/9$
     - Lily pad C: $1/9 + 1/6 + 1/3 = 11/18$
   - Briefly mention (with minimal explanation) that using matrices (whatever those are) we can compute $P^n(x, y)$ a lot easier.

5. Discuss stationary distributions. Look at lily pad and coin flip examples.

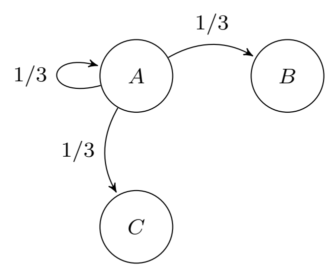
6. Speak a tiny bit about expectation, both colloquially and mathematically, as a warm up for the knights move activity.

7. Give them knight's move worksheet after setting up problem: On a standard 8×8 chess board with no pieces on it, place a knight on its normal starting square. The stochastic knight moves by selecting with uniform probability from its legal chess moves. How many moves do you expect it to take before the knight returns to its original square?

8. Discuss how Markov chains don't depend on history. Give examples and non-examples.

Worksheet #1: Markov Chains

The Set Up: Imagine you are a frog standing on a lily pad. There are two other lily pads and you hop to one *randomly.* You are equally likely to jump to any of the lily pads, including in one you're already standing on. In other words, there is a one-third (1/3) chance of jumping to each other lily pad. We represent this situation with the following diagram:



Definition: Notice that we labeled the lily pads A, B, C and D. We call the lily pads the *states* and the collection of all lily pads the *state space.*

Definition: A *probability transition function* tells us the probability that the frog hops from one state to a different state. The notation $P(x, y)$ tells us the probability the frog hops from state x to state y.

Right now we know that:

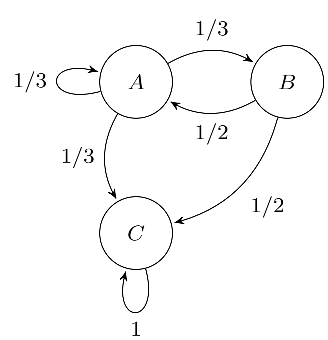
$$P(A, A) = 1/3, \qquad P(A, B) = 1/3, \qquad P(A, C) = 1/3.$$

But we aren't done. If the frog is on state B, where is it going to hop? Let's define the following probability transition functions:

$$P(B, A) = 1/2, \qquad P(B, C) = 1/2$$

$$P(C, C) = 1$$

Now we can draw a complete state diagram:

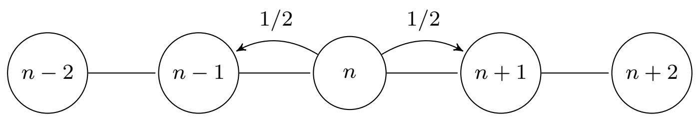


Question 1: What's the sum of the numbers on all the arrows pointing *toward* state A? What about state B? State C?

Question 2: What's the sum of the numbers on all the arrows pointing *away from* state A? What about state B? State C?

Here's a chance to draw some of your own state diagrams and probability transition functions.

Example 1 (Random walk in one-dimension): Imagine you are standing on a number line. You are equally likely to hop left or right.

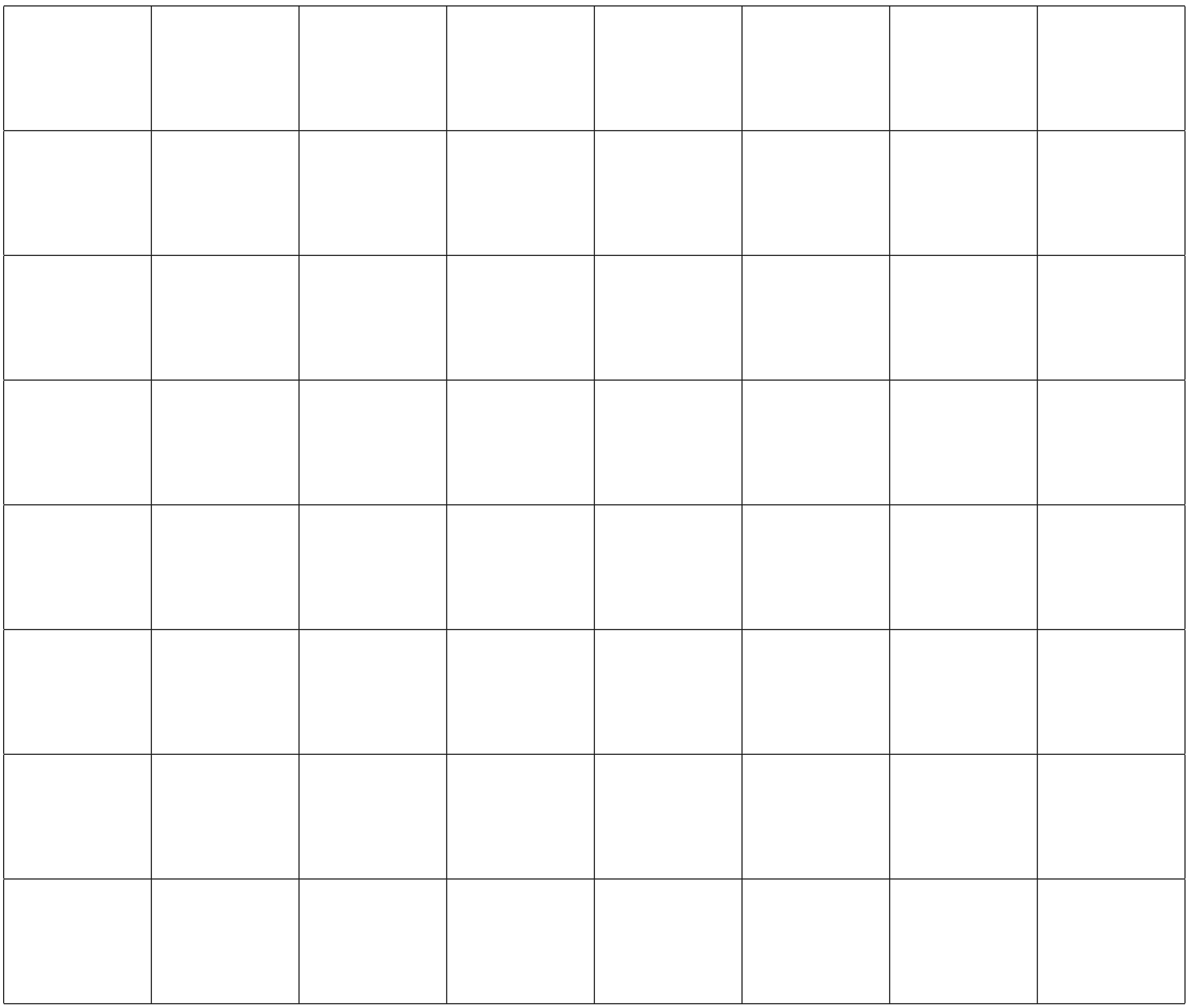


Write out the probability transition function.

Example 2 (Random walk in two-dimensions): Let's repeat the same idea in two-dimensions, on a grid. Now, you are equally likely to hop left, right, up or down. Draw the state diagram that represents a two-dimensional random walk and write the probability transition function.

Example 3 (Flipping a coin): What are the states of a coin? Draw the state diagram that represents flipping a coin and write the probability transition function.

On a standard 8×8 chess board with no pieces on it, place a knight on its normal starting square. Each time it moves the knight chooses a square at random from its legal moves. How many moves do you expect it to take before the knight returns to its original square?



Day 2: Markov Chains

1. Put up glossary words and ask for a quick review definitions: Markov chain = (state space, prob. trans. func.); $P^n(x, y)$; stationary dist.; expectation
2. Start with Worksheet #2 as warm-up/review
3. Introduce matrix multiplication; compute higher probabilities in lily pad example (if they have time/interest, mention that they can also do it for Gambler's Ruin with fixed N)
4. Gambler's Ruin: Put on your probabilist's cap. What type of questions might you want to ask? (1) Probability of reaching \$$N$ before \$0; (2) Expected time to reach one or the other.
   - (1) Let w_k =probability that you will reach fortune (N) before you lose it all (0) if you start with k dollars. We will compute w_ks for $N = 1, 2$ (trivial) then $N = 3$ by solving a system of 4 linear equations. Hopefully they get the pattern of how to solve, and I'll ask them to do it for $N = 4$. Hopefully they have noticed the pattern that $w_k = k/N$ and I can briefly explain the general proof with a system of N linear equations: $w_k = 1/2w_{k-1} + 1/2w_{k+1}$.
   - (2) Depending on the success of that activity, move up to expectation. Let $E(k)$ =expected time for game to end (reach 0 or N). Show the trivial cases: $N = 1, 2$ then think about $N = 3$. Generalize to get system of N linear equations: $E(k) = 1/2(1 + E(k-1)) + 1/2(1 + E(k+1))$ and solve to get $E(k) = k(N-k)$.
5. Two Markov chains that are interesting and might be worth mentioning:
   - Talk about random walks in higher dimensions; mention recurrence and transience
   - Card Shuffling: Talk about how we can think of the state space as all possible orderings of the card.
6. Monopoly markov chains:
   - Animation: http://www.bewersdorff-online.de/amonopoly/
   - Reference: http://www.math.uiuc.edu/ bishop/monopoly.pdf
   - Reference: http://www.tkcs-collins.com/truman/monopoly/monopoly.shtml
   - What are the Markovian and non-Markovian aspects of Monopoly? How can we use analysis of Markov chains to help us make a Monopoly strategy?

Worksheet #2: Markov Chains

Gambler's Ruin: Consider a gambler betting on coin tosses. If the coin lands on heads, she adds one dollar to her purse; if the coin lands on tails up, she loses one dollar. If she ever reaches a fortune of N dollars, she will stop playing. If her purse is ever empty, then she must stop betting.

Compute the transition probabilities and draw the state space diagram. [Hint: The states of the Markov chain should be the amount of money she has in her purse.]