# Math Explorer's Club

A Module on Combinatorics

Instructor Notes

## 1 Day 1: Pigeonhole Principle and Ramsey Theory

**Things to do days before Day 1 module:**

- Assuming there are (at most) 25 students, print out pictures of 25 cute colorful cartoon birds, paste them on posterboard, and cut out these 25 bird shapes.
- Find 25 boxes and label them 0 to 24. TIP: If there are insufficient boxes, you could partition the boxes and label each partition accordingly.

**Things to bring for Day 1 module:**

- 25 bird shapes (prepared in advance).
- 25 boxes (labeled in advance).
- 25 chess pieces (or 25 checkers chips).
- Blu-tack.
- Name tags (for students to write their names).
- Sharpies (for writing on name tags).
- Paper and writing material (pens/pencils).
- Day 1 worksheet (do not staple; pages 1-2 will be given out midway of the module, page 3 will be given at the end of the module)

**Things to do 15 minutes before Day 1 module:**

- Draw on blackboard 4 large boxes (of different shapes); use at most half the blackboard.
- Lay out bird shapes, boxes, chess pieces, and Blu-tack on table.

**Overview of Day 1 module:**

1. **Beginning Activity/Ice-breaker:** Tell students to do the following.
   - Choose a bird, and then place the chosen bird in any of the shapes on the blackboard using Blu-tack.
   - Go around the classroom, shake hands with other students, and introduce your name. Keep track of how many students you have shaken hands with.
   - When the instructor says "Time's up!" take one chess piece, and place it in the box labeled with the same number as the number of students you have shaken hands with.
2. Introduce pigeonhole principle with a simple example of 7 pigeons resting in 6 pigeonholes.
   - Claim: There is at least one pigeonhole with at least 2 pigeons.
   - The pigeons could try to squeeze together, so maybe there is more than one pigeonhole with at least 2 pigeons.

- Maybe the pigeons decide to rearrange themselves, so that there are no pigeonholes with exactly 2 pigeons.
- But no matter how the pigeons arrange themselves, there will always be **at least** one pigeonhole with **at least** 2 pigeons.

3. State the pigeonhole principle (simplest version): If there are $n+1$ pigeons resting in $n$ pigeonholes, then at least one pigeonhole has at least 2 pigeons.

4. Next, state the pigeonhole principle (simple version): If there are $kn+1$ pigeons resting in $n$ pigeonholes, then at least one pigeonhole has at least $k+1$ pigeons.

   - Give concrete example: If there are $19 = 3 \times 6 + 1$ pigeons and 6 pigeonholes, then at least one pigeonhole has at least $4 = 3 + 1$ pigeons.

5. Get students to start thinking about a more general version of the pigeonhole principle.

   - Question: What if there are $n$ pigeonholes, but the number of pigeons is not of the form $kn+1$?
   - For example, what if there are 22 pigeons and 6 pigeonholes?
   - What if there are 12 pigeons and 6 pigeonholes?
   - What about 600 pigeons and 6 (very large) pigeonholes?
   - Can we find a **general version** of the pigeonhole principle?

6. As students start thinking about what a more general version of the pigeonhole principle should look like, introduce the notion of the floor and ceiling of numbers.

   - Draw a picture of a building (both underground and above ground) on blackboard. (Do not remove the birds that are still on the blackboard.) Explain the floor/ceiling function using the analogy of the floors and ceilings of the building. Ask students to volunteer a few numbers (preferably not integers), and work out the floors and ceilings of these numbers.

7. Group Discussion: Now that the students understand the floor/ceiling function, ask them again what a more general version of the pigeonhole principle should look like. Remind them of the birds on the blackboard.

   - Prompt: What are the "pigeons" and "pigeonholes" on the blackboard?

8. Reveal the pigeonhole principle (general version): If there are $m$ pigeons in $n$ pigeonholes, then at least one pigeonhole has at least $\lceil \frac{m}{n} \rceil$ pigeons.

   - Give concrete example: If there are 22 pigeons and 6 pigeonholes, then at least one pigeonhole has at least $\lceil \frac{22}{6} \rceil = \lceil 3.666\ldots \rceil = 4$ pigeons.

9. Remind students that the "pigeons" can be whatever objects they choose. Similarly, the "pigeonholes" can have whatever "labels" they want, such as "Cute Blue Birds Only".

10. **NYC bald people**: Explain to students the following implication of the pigeonhole principle.

    - **Claim:** There are at least five, non-bald, people in New York City with exactly the same number of hairs on their heads.
    - Fact: Population of NYC $\approx 8.5$ million
    - Fact: Research has shown that a person has on average about $150,000$ hairs.
    - Most people in NYC are not bald.
    - We can safely assume that at least 4,000,001 people are not bald.
    - We can also assume that a person has at most 1,000,000 hairs.
    - "Pigeons" = non-bald people in NYC. "Pigeonholes" are labeled with a number from 1 to $1,000,000$, representing the number of hairs.

- By the Pigeonhole Principle, there are at least $\lceil\frac{4{,}000{,}001}{1{,}000{,}000}\rceil = 5$ non-bald people in NYC with the same number of hairs (i.e. there are at least 5 "pigeons" in the same "pigeonhole").

11. **Shaking Hands:** Let students think about the following scenario: 25 students in a classrom went around shaking hands. Can you always find 2 students who have shaken hands with the same number of people?
    - Question: How is this related to the Pigeonhole Principle?
    - Question: What are the "pigeons", and what are the "pigeonholes"?
    - Remind the students that they went around shaking hands at the beginning of the module.
    - Prompt: Did everyone shake hands with at least one person? Did anyone shake hands with everyone else?
    - Prompt: What are the "pigeons" and what are the "pigeonholes"? What would be suitable "labels" for the "pigeonholes"? Use the labeled boxes and chess pieces to explain.
    - If the students do not see the relation, encourage them by saying that sometimes, the "pigeons" and "pigeonholes" are not immediately obvious.
12. **Problem Solving Session:** Hand out pages 1-2 of Day 1 worksheet. Let students try out the problems at the bottom of page 2. Go around to check on progress and give hints if necessary.
13. **Friends and Strangers:** Take 6 bird shapes and arrange them in a circular pattern on the blackboard. Choose one of the birds and name it, e.g. Tweety.
    - Question: Among the 6 birds, can we always find 3 of them that are either all friends with each other or all strangers with each other?
    - Between every pair of birds, we can draw a **blue** edge for **friends**, or a **red** edge for **strangers**.
    - Let's look at the edges incident to Tweety.
      There are 5 other birds that Tweety can be friends or strangers with.
    - We have no idea who are friends or strangers with Tweety.
    - But by the Pigeonhole Principle, at least 3 of these 5 edges have the same color.
    - Suppose 3 of these 5 edges are colored blue. Look at the 3 neighbors of Tweety joined by blue edges.
    - Prompt: What happens when any two of them are friends? What happens when no two of them are friends?
    - Remember that it doesn't matter which bird we begin with. Whichever bird we choose, the Pigeonhole principle tells us that this bird will always have 3 edges coming out of it with the same color.
    - New Question: Among 5 birds, can we always find 3 of them that are either all friends with each other or all strangers with each other?
    - Prompt: Can we always find 3 of these 5 birds that are either all friends with each other or all strangers with each other?
    - Claim: Answer is no (Hint: star within pentagon).
14. Recap: In any group of birds, if the total number of birds is **at least 6**, then we can **always** find either 3 of them who are all friends with each other, or 3 of them who are all strangers with each other.
    - This **minimum** total number of birds for always finding **3** mutual friends or **3** mutual strangers is written as $R(\mathbf{3}, \mathbf{3})$.
    - In other words, we have shown that $R(3, 3) = 6$.
    - Question: What about having 4 of them who are all friends with each other? Or 5 of them who are all strangers with each other?

15. **Ramsey Theory:** State Ramsey's theorem: Let $r$ and $s$ be any positive integers. In a **large enough** community of people, no matter who are friends or who are strangers, there will always be either $r$ people who are all friends with each other, or $s$ people who are all strangers with each other.

    - You may need to have a very very large number of people to guarantee that you can always find $r$ mutual friends, or $s$ mutual strangers.
    - The minimum total number of people in the community for this to be always true is called a **Ramsey number**, and it is written as $R(r, s)$.

16. In addition to $R(3, 3) = 6$, there are other known exact values.

    - $R(3, 4) = 9$
    - $R(4, 4) = 18$ (proven in 1979)
    - $R(4, 5) = 25$ (proven in 1995)
    - $R(5, 5) =??$ (unknown)
    - $R(6, 6) =??$ (unknown)

17. Some known inequalities:

    - $43 \leq R(5, 5) \leq 49$.
    - $102 \leq R(6, 6) \leq 165$.

18. **Quote of the Day:** "Imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, we should attempt to destroy the aliens." (by Paul Erdős)

19. Teaser for Day 2: Tell students that they will be exploring some combinatorics that involves fighting dragons! Give out page 3 of Day 1 worksheet (which summarizes the Ramsey theory part). End of Day 1 module.

# 2 Day 2: Invariance Principle and Extremal Principle

**Things to bring for Day 2 module:**

- 25 bird shapes.
- 5 checkers boards (or any 5 large 8x8 boards).
- 5 sets of poker chips (at least 5x64 poker chips would be ideal).
- Blu-tack.
- Name tags (for students to write their names).
- Sharpies (for writing on name tags).
- Paper and writing material (pens/pencils).
- Day 2 worksheet (staple all pages together)

**Things to do 15 minutes before Day 2 module:**

- Write on the blackboard the following string of letters: MECMECMECM

**Overview of Day 2 module:**

1. Welcome back to the **M**ath **E**xplorer's **C**lub!

2. **MEC Game:** Rules of the game: Our little blue friend  will start by thinking of a long string of letters using only **M**, **E**, and **C**. Once we are given this string of letters, we are allowed to do any of the following three moves:

   - Strike off the letters **M** and **E**, then write the letter **C**.
   - Strike off the letters **E** and **C**, then write the letter **M**.
   - Strike off the letters **C** and **M**, then write the letter **E**.

   Can we find a sequence of moves such that exactly one and only one letter remains? Can this letter be **M**? What about **E**? What about **C**?

3. Our little blue friend  is thinking of the string **MECMECMECM**.

4. Give a concrete example of how to play the MEC Game. (Note: After each move, the number of remaining letters drop by one.)

5. Let the students try out the MEC game by themselves (distribute paper and writing materials). After some time, invite students who have won the game (i.e. found a sequence of moves such that exactly one and only one letter remains) to show their winning strategy on the blackboard. Tell them to write out every step (no erases). [Note to instructor: The game can only be won with a single letter **M** remaining.]

6. Can you win the MEC game?

   - Let's look at the string **MECMECMECM** that our little friend  gave us.
   - At each step in the MEC game, let **m** be the number of **M**'s, let **e** be the number of **E**'s, and let **c** be the number of **C**'s.
   - So, at the beginning of our game, we have $\mathbf{m} = 4$, $\mathbf{e} = 3$, $\mathbf{c} = 3$.
   - Question: Will the numbers **m**, **e**, and **c** be odd or even after each move?
   - Answer: Odd becomes even, and even becomes odd.
   - Key Idea: At the start, both **e** and **c** are odd, so after any sequence of moves, **e** and **c** will either be both odd, or both even. Similarly, for **m** and **e**, one number is odd, the other is even, so after any sequence of moves, one of them must be odd, and the other must be even.
   - Conclusion: We can never end the game with a single letter **E** or **C**.
   - Did any of the students claim to have ended the game with **E** or **C**?

7. **The Invariance Principle:** Invariance is a simple idea in combinatorics.

   - Suppose we have a "process" (e.g. the MEC game) that involves repeated "moves". There could be several different possible "moves".
   - If "something" does not change after every possible "move", then this same "something" **will never change after any sequence of "moves"**. This "something" could be any "property" or "value", and we call it an **invariant**.
   - (Note: In mathematics, the word 'invariant' can be either a noun or an adjective.)

8. Example: If you are cutting a cake into several pieces, then however many pieces you cut it into, the total volume of all the pieces remains unchanged.

   - Prompt: What is an invariant in our cake-cutting example?

9. Remark to students: Invariants may not be obvious. But if there is some kind of repetition, look for what does not change!

10. **Remainders after Division:** Teach students the basics of modular arithmetic.
    - Remark: Before we look at more complicated examples of invariance, let's look at remainders after division.
    - If $r$ is the remainder when $n$ is divided by $m$, then we say "$n$ is congruent to $r$ modulo $m$", and we write "$n \equiv r \pmod{m}$". (Do some concrete examples.)
11. **Fighting Dragons:** Dragon Puzzle: A dragon has 100 heads initially. A knight can cut off 15, 17, 31 or 5 heads each time with one blow of his sword. In each of the cases, 24, 2, 13, or 17 new heads grow on the dragon's shoulders. If all the heads are cut off, the dragon dies. Can the dragon ever be killed by our knight?
12. Class Discussion: Let students discuss whether the dragon can be killed.
    - Prompt: What is the change in the number of heads in each case? What is this change modulo 3? What does not change?
13. **The Upside Down Pigeon Puzzle:** Draw a 4-by-4 rectangular grid big enough to fit 16 birds. Paste 16 birds (using Blu-tack) on this grid in the configuration shown in the Day 2 worksheet.
    - There are 16 pigeons resting in 16 pigeonholes as shown on the blackboard. One of them is upside down. You can switch the orientation of all the pigeons in a row, column, or any line parallel to one of the diagonals. You can also switch the orientation of any of the 4 corner pigeons. Right-side up pigeons become upside down, and upside down pigeons become right-side up. Is there a sequence of switches so that all the pigeons become right-side up?
14. Conduct class discussion on the Upside Down Pigeon puzzle.
    - Prompt: Focus on the boundary squares minus the corners (top 2, bottom 2, left 2, right 2). What remains invariant?
15. **Checkerboard Puzzle:** Divide the students into 5 groups, with each group having 1 checkerboard and 1 set of poker chips.
    - Four chips are placed at the bottom left part of an $8 \times 8$ checkerboard in a $2 \times 2$ area. We are allowed the following move: If a square has a chip, while the squares above and to the right of it are empty, then we can remove the chip and place new chips on each of the other two squares. Can you find a sequence of moves such that the bottom left $2 \times 2$ area no longer have chips?
16. The checkerboard puzzle should be the most fun activity of the day. Let the students try out the puzzle. You should expect none of them to figue out a proof that no such sequence of moves is possible. After some time, reveal the key idea.
17. Key Idea: Assign weights to the squares. Explain the details outlined in Day 2 worksheet. Teach students how to compute geometric series. (This should tie in with the geometric series taught in high school for some of the more advanced students.)
    - Question: What happens to the weight every time we make a move?
18. Remark to students: From the Checkerboard Puzzle, we learned that sometimes we can assign weights and find an invariant based on these weights. It's amazing how a simple idea like **"things that don't change will remain unchanged"** can have very non-obvious implications.
19. Remark to students: Let's look at another simple idea that also has very non-obvious implications.
20. **Non-intersecting Lines:** Twenty points on a plane
    - There are 20 points on a plane, where no 3 of them are collinear. 10 of them are colored red, and the other 10 are colored blue. Can we pair up the red points with the blue points such that if we draw a straight line segment joining the two points (one red and one blue) in each of the 10 pairs, then none of the 10 line segments intersect? Can we always do this?

- With 20 points, there are many many ways to pair up the red points with the blue points. (In fact there are 3628800 possible ways.)
- BUT the total number of possible pairings, no matter how large, is **finite**!

21. Key Idea: By the **extremal principle**, among all these finitely many possible pairings, there is one with a **smallest** total sum of all the lengths of the line segments. This pairing cannot have any intersecting line segments!

22. **Extremal Principle:**

    - The extremal principle is based on the following easy fact: Every **finite** collection of integers or real numbers has a smallest element and a largest element. It is possible to have more than one smallest or largest element if repetitions are allowed.
    - Infinite collections can be complicated, but the following is still true: Every (possibly infinite) collection of **positive** integers has a smallest element. This is called the **well-ordering principle**.
    - Warning: The well-ordering principle is not true for real numbers.
      Example: $0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001, \ldots$.

23. State the extremal principle in two parts:

    - If we assign integers or real numbers to a **finite** collection of objects, then there is an object with the smallest assigned number, and there is an object with the largest assigned number.
    - If we assign **positive** integers to a collection of objects, then there is an object with the smallest assigned number.

24. **One-way Road Puzzle:** There is a country far far away, where every road is one-way. Every pair of cities is connected by exactly one direct road. Show that there exists a city which can be reached from every city directly or via at most one other city.

    - Note: There is probably not enough time to let the students try this puzzle. Instead, you can let this puzzle be another example of the extremal principle that the students can think about after the module.
    - Use the bird shapes to depict cities.
    - First Key Idea: There can only be a **finite** number of cities, so by the extremal principle, there is some city (maybe more than one) with the maximum number of direct roads leading into it.
    - Second Key Idea: We claim that any such city can be reached from every city directly or via at most one other city. (Explain why. Suppose not.. What does that mean? Get a contradiction.)
    - **Remember:** Every pair of cities is connected by **exactly one** direct road.

25. Remark to students: Sometimes, if something sounds improbable, maybe it is possible in an extremal case. Or maybe it is impossible because of an invariant.

26. **Problem Solving Session**: Hand out Day 2 worksheet (all pages). Let students try out the problems at the bottom of page 4. Go around to check on progress and give hints if necessary.

27. **Infection**: Ask students to look at the challenge question in the worksheet. Draw an 8-by-8 rectangular grid on the blackboard. Get 7 volunteers to choose 7 different squares as initial infected squares. Let the infection spread by shading squares to indicate infection. Allow the students to "see and understand" why the infection does not spread to the whole 8-by-8 grid. Try again with a new set of 7 squares, this time challenging the students to choose the 7 squares wisely so that the infection is maximized. Discuss with the students possible reasons why the infection does not spread to the whole grid.

    - Look at the perimeter of all the infected squares. Can this perimeter increase? What is the maximum perimeter of the initial 7 infected squares?

28. **Summary**: So... What is Combinatorics?

- The study of finite or countable discrete structures? (Wikipedia)
- We now know four different ideas in combinatorics!
  - Pigeonhole Principle: If $n+1$ things are in $n$ boxes, then some box has at least 2 things.
  - Ramsey Theory: A large enough system, no matter how disorderly, has some structure.
  - Invariance Principle: Things that don't change will remain unchanged.
  - Extremal Principle: A finite list of numbers, no matter how long, has a smallest or largest element.
- Remark to students: These are just four of the many different ideas in combinatorics, but they give a good flavor of what combinatorics is like.
- Question: What do you think combinatorics is?