Math Explorer's Club

A Module on Combinatorics

Instructor: Ernest Chong

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1 Pigeonhole Principle

Imagine there are 7 pigeons resting in 6 pigeonholes. Will you always be able to find a pigeonhole with at least 2 pigeons? The pigeons could try to squeeze together, so maybe there is more than one pigeonhole with at least 2 pigeons. Maybe the pigeons decide to rearrange themselves, so that there are no pigeonholes with exactly 2 pigeons. But no matter how the pigeons arrange themselves, there will always be **at least** one pigeonhole with **at least** two pigeons. This observation captures an idea that mathematicians call the "pigeonhole principle".

There are several versions of the pigeonhole principle.

Pigeonhole Principle (simplest version)
If there are $n+1$ pigeons resting in n pigeonholes, then at least one pigeonhole has at least 2 pigeons.

Pigeonhole Principle (simple version).
If there are $kn+1$ pigeons resting in n pigeonholes, then at least one pigeonhole has at least $k+1$ pigeons.

There is a more general version of the pigeonhole principle, where we can have any number of pigeons. To state this general version, we will need the idea of the floor function and the ceiling function.

- Given any real number x, the **floor** of x, which we write as $\lfloor x \rfloor$, is the largest integer not strictly greater than x.
- Given any real number x, the **ceiling** of x, which we write as $\lceil x \rceil$, is the smallest integer not strictly less than x.

To understand what $\lfloor x \rfloor$ and $\lceil x \rceil$ mean, think of the number line as a building, where number 0 is at the ground level, which we call level 0. The positive integers $1, 2, 3, \dots$ are above the ground at levels $1, 2, 3, \dots$ respectively, while the negative integers $-1, -2, -3, \dots$ are underground at levels $-1, -2, -3$ respectively. In this building, the levels are labeled only by integers. Levels can be thought of as both the floor of the space above, and the ceiling of the space below. So if n is an integer, then $\lfloor n \rfloor = \lceil n \rceil = n$.

What about a number that is not an integer? For example, $\pi \approx 3.14159\dots$ is in between level 3 and level 4, so we can think of the floor of π as 3, and the ceiling of π as 4. Consequently, $\lfloor \pi \rfloor = 3$ and $\lceil \pi \rceil = 4$.

What about negative numbers? For example, $-\pi \approx -3.14159\dots$ is in between level -4 and level -3, so we can think of the floor of $-\pi$ as -4, and the ceiling of $-\pi$ as -3. Thus $\lfloor -\pi \rfloor = -4$ and $\lceil -\pi \rceil = -3$.

Historical Note: Before the symbols $\lfloor x \rfloor$ and $\lceil x \rceil$ became popular, the floor of x was written as $[x]$. Some textbooks still write $[x]$ for the floor function, but there is the risk of it being mistaken for the usual square brackets. To avoid confusion, use $\lfloor x \rfloor$ and $\lceil x \rceil$. They look cool anyway.

We are now ready to state the general version of the pigeonhole principle.

Pigeonhole Principle (general version).
Let m, n be positive integers. If there are m pigeons resting in n pigeonholes, then at least one pigeonhole has at least $\lceil \frac{m}{n} \rceil$ pigeons.

Of course, we can always replace "pigeons" by other objects, and "pigeonholes" by "boxes" (perhaps with whatever labels you want). The idea is still the same. If there are m objects in n boxes, then at least one box has at least $\lceil \frac{m}{n} \rceil$ objects.

Example 1 (Hairs on heads).
The pigeonhole principle implies the following (surprising?) fact: There are at least five non-bald people in New York City with exactly the same number of hairs on their heads. To see why, note that the latest population census says New York City has a population of around 8.5 million, while recent medical research shows that a person has on average about $150,000$ hairs. If you have been to NYC, you would notice that most of the people there are not bald. We can safely assume that at least $4,000,001$ people are not bald. We can assume that a person can have at most $1,000,000$ hairs. Thus, by the pigeonhole principle, there are at least $\lceil \frac{4{,}000{,}001}{1{,}000{,}000} \rceil = 5$ non-bald people in NYC with the same number of hairs. We have no idea who they are, but we can be sure that they exist.

Example 2 (Shaking hands).
25 students went around shaking hands. No matter whose hands were shaken, there are at least 2 students who have shaken hands with the same number of people. Observe that every student must shake hands with at least 0 students, and can shake hands with at most 24 students. So we have 25 pigeonholes labeled by $0, 1, \ldots, 24$, corresponding to the number of people a student has shaken hands with. If some student has shaken hands with 0 students, then no other student could possibly have shaken hands with 24 people. Similarly, if some student has shaken hands with 24 students, then no other student could possibly have shaken hands with 0 students. Hence, the possible pigeonholes are really either labeled $0, 1, \ldots, 23$ or labeled $1, 2, \ldots, 24$, a total of 24 pigeonholes. There are 25 students (pigeons), so by the pigeonhole principle, at least two students has shaken hands with the same number of people.

Useful Tip: When you want to apply the pigeonhole principle, the "pigeons" and the "pigeonholes" are sometimes not obvious. So ask yourself, what could the "pigeons" be? What would be suitable "labels" for the "pigeonholes"? Remember, "pigeonholes" can come in different shapes and sizes, so you can even have a "small pigeonhole" with a small maximum number of "pigeons" allowed.

2 Problems on Pigeonhole Principle

1. There are 10 red socks, 14 green socks, and 20 blue socks in a drawer. The room is dark. How many socks must you take out of the drawer to be sure of having a pair of socks of the same color?

2. Let S be a collection of 11 different integers between 1 and 20 inclusive.

   (a) Show that we can always find two consecutive integers in S.

   (b) Show that we can always find two different integers in S whose sum is odd.

   (c) Show that there are two different integers a and b in S, such that a divides b.

3. Given any 5 different points on the surface of a sphere, show that we can find a closed hemisphere which contains at least 4 of them. (A hemisphere is closed if it includes all points on the boundary.)
   [Hint: Pigeonholes can come in any shape and size.]

4. Suppose your computer has randomly generated 10 integers (possibly with repetitions) between 0 and 100 inclusive. Can you always choose some of these 10 integers in two different ways, but with the same sum?
   [Example: If the computer generated the integers $2, 3, 5, 7, 11, 13, 49, 49, 79, 100$*, then you can choose* $2 + 49 + 49 = 100$*, or you can choose* $2 + 3 + 5 + 7 + 11 + 13 + 49 = 11 + 79$*.]*

5. During his free time, John wrote down 100 integers (possibly with repetitions) on a piece of paper. Is it true that no matter what these 100 integers are, you can always find some (at least one) of these integers whose sum is divisible by 100?

3 Ramsey Theory

Ramsey theory is another important branch of combinatorics that was popularized by the famous Hungarian mathematician Paul Erdős (pronounced *air-dish*). To get a feel for what Ramsey theory is about, let's look at the following example.

Example (Friends and Strangers).
In a group of 6 people, some of them are friends, some of them are strangers. We have no idea who are friends, and who are strangers. But Ramsey theory says that no matter what, we can always find either 3 of them who are all friends with each other, or 3 of them who are all strangers with each other.

Why is this true? Choose a random person among these 6 people. Suppose this person is called Emily. Emily can be friends or strangers with the remaining 5 people. For each of these 5 people, imagine that we place a blue sticker on the person's forehead if he or she is friends with Emily, and we place a red sticker on the person's forehead if he or she is strangers with Emily. By the pigeonhole principle, at least 3 of these 5 people have stickers of the same color. Suppose the color is blue, which means that Emily is friends with at least 3 of these 5 people. If at least 3 of them have red stickers, we could always switch the colors red and blue, and replace 'friends' with 'strangers' (and vice versa) in the rest of the argument. Let's suppose Amy, Bob and Cathy are three of Emily's friends among these 5 people. If any two of Amy, Bob and Cathy are friends with each other, say for example Amy and Bob, then Amy, Bob and Emily are all friends with each other. If none of Amy, Bob, Cathy are friends with each other, then Amy, Bob and Cathy are all strangers with each other. Either way, we can always find 3 of them who are either all friends with each other, or all strangers with each other.

Question: Is it possible to have a group of 5 people, such that no 3 of them are all friends with each other, and no 3 of them are all strangers with each other? *[Hint: Imagine a star inscribed in a pentagon.]*

Ramsey's Theorem
Let r and s be any positive integers. In a **large enough** community of people, no matter who are friends or who are strangers, there will always be either r people who are all friends with each other, or s people who are all strangers with each other.

- You may need to have a very very large number of people to guarantee that you can always find r mutual friends, or s mutual strangers.
- The minimum total number of people in the community for this to be always true is called a **Ramsey number**, and it is written as $R(r, s)$.

The example above tells us that $R(3, 3) = 6$.

Roughly speaking, Ramsey's theorem can be thought of as saying that a large enough system, no matter how disorderly, has some structure. You may think that many (if not all) values for $R(r, s)$ are known. Unfortunately, this is not the case. For example, the exact values of $R(5, 5)$ and $R(6, 6)$ are not known. However, mathematicians have found fairly good estimates for these values: $43 \leq R(5, 5) \leq 49$ and $102 \leq R(6, 6) \leq 165$. The following quote by Paul Erdős tells us that finding the exact values of Ramsey numbers is in general very difficult:

> *"Imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of $R(5, 5)$ or they will destroy our planet. In that case, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for $R(6, 6)$. In that case, we should attempt to destroy the aliens."* – Paul Erdős