Math Explorer's Club

A Module on Combinatorics

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1 Invariance Principle

Activity: The MEC Game.
Our little blue friend  will start by thinking of a long string of letters using only M, E, and C. Once we are given this string of letters, we are allowed to do any of the following three moves:

- Strike off the letters M and E, then write the letter C.
- Strike off the letters E and C, then write the letter M.
- Strike off the letters C and M, then write the letter E.

Can we find a sequence of moves such that exactly one and only one letter remains? Can this letter be M? What about E? What about C?

Example:  is thinking of the string "MECMECMECM".

The MEC Game is a good example to learn about invariance. Invariance is a simple idea in combinatorics. Suppose we have a "process" (e.g. the MEC game) that involves repeated "moves". There could be several different possible "moves". If "something" does not change after every possible "move", then this same "something" *will never change after any sequence of "moves"*. This "something" could be any "property" or "value", and we call it an **invariant**. (Note: In mathematics, the word 'invariant' can be either a noun or an adjective.)

Example: If you are cutting a cake into several pieces, then however many pieces you cut it into, the total volume of all the pieces remains unchanged. Hence, the total volume of all the pieces of cake is an invariant for this cake-cutting process.

An invariant in the MEC Game.

Let's look at the string MECMECMECM that our little friend  gave us. At each step in the MEC game, let m be the number of M's, let e be the number of E's, and let c be the number of C's. So, at the beginning of our game, we have $m = 4$, $e = 3$, and $c = 3$.

There are three possible moves in the MEC Game. At any time in the game, if **m** is even, then after any of the three possible moves, the number **m** will become odd. If instead **m** is odd, then after any of the three possible moves, the number **m** will become even. Similarly, if **e** is even or odd, then after any move, **e** will become odd or even respectively. Likewise for **c**. Hence, for the three numbers **m**, **e**, and **c**, odd numbers will become even and even numbers will become odd, each time we make a move.

At the start of the game, e and c are both odd, so after any sequence of moves, e and c will either be both odd, or both even. If there is exactly one E (or C) at the end of the game, then since 1 is odd, we infer that e and c must both be odd, so there must be at least one C (or E). Thus, we conclude that we can never end the game with a single letter E or C.

Similarly, at the start of the game, $m = 4$ is even and $e = 3$ is odd, so after any sequence of moves, one of m and c must be even, and the remaining number must be odd. We already know that e and c are either both even or both odd, so we conclude that if we can end the game with exactly one letter remaining, then this letter must be M! Can you find a sequence of moves until only the letter M remains?

Before we look at more complicated examples of invariance, let's look at remainders after division. We know that an integer is **even** if it is divisible by 2, and an integer is **odd** if it is not divisible by 2. Another way of determining whether an integer is even or odd is to look at its remainder when divided by 2. Every integer, when divided by 2 has a remainder of either 0 or 1. If the remainder is 0, then the integer is even. If the remainder is 1, then the integer is odd.

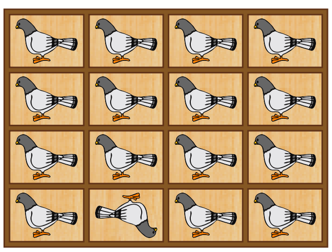
More generally, when we divide an integer n by some positive integer m, then its possible remainders are $0, 1, \ldots, m-1$. If r is the remainder after dividing by m, then we say "n is congruent to r modulo m", and we write "$n \equiv r \pmod{m}$". This expression "$n \equiv r \pmod{m}$" means that n and r have **the same remainder** when divided by m. For example, $22 \equiv 1 \pmod{3}$, $100 \equiv 0 \pmod{5}$, $7 \equiv 7 \pmod{13}$, and $25 \equiv 15 \pmod{10}$.

Activity: Dragon Puzzle.
A dragon has 100 heads initially. A knight can cut off 15, 17, 31 or 5 heads each time with one blow of his sword. In each of the cases, 24, 2, 13, or 17 new heads grow on the dragon's shoulders. If all the heads are cut off, the dragon dies. Can the dragon ever be killed by our knight?

Question: In the dragon puzzle, can you find an invariant?

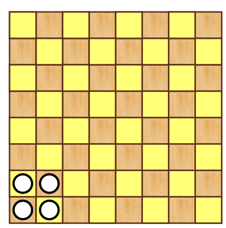
Answer: Observe that no matter how many heads the knight cuts off, the total number of heads (after the dragon regrows some heads) will always be $\equiv 1 \pmod{3}$. The dragon is killed when it has 0 heads. However, since $0 \not\equiv 1 \pmod{3}$, our knight can never kill the dragon.

Activity: The Upside Down Pigeon Puzzle.

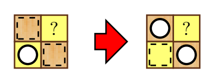


There are 16 pigeons resting in 16 pigeonholes as shown on the left. One of them is upside down. You can switch the orientation of all the pigeons in a row, column, or any line parallel to one of the diagonals. You can also switch the orientation of any of the 4 corner pigeons. Right-side up pigeons become upside down, and upside down pigeons become right-side up. Is there a sequence of switches so that all the pigeons become right-side up?

Activity: The Checkerboard Puzzle.
Four chips are placed at the bottom left part of an 8×8 checkerboard in a 2×2 area as shown on the right. We are allowed the following move: If a square has a chip, while the squares above and to the right of it are empty, then we can remove the chip and place new chips on each of the other two squares. Can you find a sequence of moves such that the bottom left 2×2 area no longer have chips?



Allowed Move



Square with '?' can be empty or non-empty.

An invariant in the Checkerboard Puzzle.

Key Idea: Assign weights to the squares.
Starting from the bottom left corner of the checkerboard, number the rows (rightward) and columns (upward) by $1, \ldots, 8$, so that the four squares in the bottom left 2×2 area have coordinates $(1,1)$, $(2,1)$, $(1,2)$, $(2,2)$. Assign each square (a,b) with the weight $\frac{1}{2^{a+b-1}}$ as shown on the left. Define the total weight of all the chips to be the sum of the weights of the squares that the chips occupy.

Notice that after every move, the total weight of all the chips remains invariant. The total weight of the 4 chips given at the start is $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} = 2\frac{1}{4}$. Thus, the total weight after any sequence of moves will always be $2\frac{1}{4}$. If the bottom left 2×2 area no longer have chips, then the total weight is at most $1\frac{3}{4}$ (Why?), which is impossible.

2 Extremal Principle

Before we introduce the extremal principle, we first look at the following easy fact:

- Every **finite** non-empty collection of integers or real numbers has a smallest element and a largest element. It is possible to have more than one smallest or largest element if repetitions are allowed.

Whenever we say a "non-empty" collection of things (where "things" could be "numbers", "objects", or "cute birds"), it means that the collection has at least one thing (e.g. number, object, or cute bird).

Infinite collections can be complicated, but the following is still true:

- Every (possibly infinite) non-empty set of **positive** integers has a smallest element. This is called the **well-ordering principle**.

Warning: The well-ordering principle is *not true* for real numbers. For example, the following infinite sequence of positive numbers has no smallest element.

$$0.1, \quad 0.01, \quad 0.001, \quad 0.0001, \quad 0.00001, \quad 0.000001, \ldots$$

The **extremal principle** has two parts, corresponding to the above two facts.

- If we assign integers or real numbers to a non-empty **finite** collection of objects, then there is an object with the smallest assigned number, and there is an object with the largest assigned number.
- If we assign **positive** integers to a non-empty collection of objects, then there is an object with the smallest assigned number.

Example 1 (Non-intersecting Lines).
There are $2n$ points on a plane, where no 3 of them are collinear. n of them are colored red, and the other n are colored blue. No matter what n is, we claim that we can always pair up the red points with the blue points, such that if we draw a straight line segment joining the two points (one red and one blue) in each of the n pairs, then none of the n line segments intersect.

It is clear that when n is large, there are many many ways to pair up the red points with the blue points. But the total number of possible pairings, no matter how large, is **finite**! Thus, by the extremal principle, among all these finitely many possible pairings, there is one with a **smallest** total sum of all the lengths of the line segments. This pairing cannot have any any intersecting line segments. This is because if there is a pair of intersecting line segments, then we can always replace this pair with another pair whose total length is strictly smaller, and for a pairing with the smallest total length, this cannot happen.

Example 2 (One-way Road Puzzle).
There is a country far far away, where every road is one-way. Every pair of cities is connected by exactly one direct road. Then, there exists a city which can be reached from every city directly or via at most one other city.

To show why this is true, we again use the extremal principle. First, we note that there can only be a **finite** number of cities, so by the extremal principle, there is some city (maybe more than one) with the **maximum number** of direct roads leading into it. Let X be any such city, and let m be this maximum number of direct roads leading into it. We claim that X can be reached from every city directly or via at most one other city. Let $Y_1, \ldots, Y_m$ be all the cities that have direct roads leading into X. Suppose on the contrary that there is some city Z distinct from $X, Y_1, \ldots, Y_m$ such that X cannot be reached from Z via at most one other city. This means that there is no direct road from Z to each of $Y_1, \ldots, Y_m$, otherwise X can be reached from Z via one other city. Also, since Z is not any of the cities $Y_1, \ldots, Y_m$ (which are all the cities with direct roads leading into X), we know that X cannot be reached from Z directly.

Now, we are told that every pair of cities is connected exactly by one direct road. Thus, there must be a direct road from X to Z. Similarly, since there is no direct road from Z to each of $Y_1, \ldots, Y_m$, there must be a direct road from each of $Y_1, \ldots, Y_m$ to Z. Consequently, there are at least $m + 1$ cities $(X, Y_1, \ldots, Y_m)$ with direct roads leading into Z, which contradicts the assumption that X has the maximum number of direct roads leading into it. Therefore, no such city Z exists, and we conclude that X can be reached from every city directly or via at most one other city, as claimed.

3 Problems on Invariance Principle and Extremal Principle

Useful Tip: Sometimes, if something sounds improbable, maybe it is possible in an extremal case. Or maybe it is impossible because of an invariant.

1. Seven of the eight corners of a unit cube are labeled with '+', and the remaining eighth corner is labeled with '−'. We are allowed to choose any edge of the cube and switch the signs of the labels of the two corners belonging to this edge. Is there a sequence of switches such that the labels of all the corners become '−'?

2. There are five A's and five B's written on a blackboard. You may erase any two letters, then write A if they are equal and write B if they are unequal. Do this until there is only one letter remaining on the blackboard. Is it A or B? Could it be either A or B, depending on how you erased the letter? Or is this last remaining letter always the same?

3. There is a town with 100 people. Every person in this town has at most 3 enemies. Can you always divide this town into two groups (not necessarily of the same size) such that every person has at most one enemy in his or her group?
(We assume that being enemies is symmetric, so if Tom is an enemy of Jerry, then Jerry is an enemy of Tom.)

Challenge Question: Seven 1×1 cells of an 8×8 square are infected. In one time unit, the cells with at least two infected neighbors (having a common side) become infected. Can the infection spread to the whole 8×8 square? Why or why not?

4 Summary of Module on Combinatorics

Over two weeks, we have learned four different ideas in combinatorics:

- **Pigeonhole Principle:** If $n + 1$ things are in n boxes, then some box has at least 2 things.
- **Ramsey Theory:** A large enough system, no matter how disorderly, has some structure.
- **Invariance Principle:** Things that don't change will remain unchanged.
- **Extremal Principle:** A finite list of numbers, no matter how long, has a smallest or largest element.