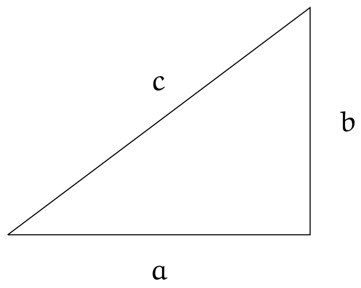
1 Pythagoras' Theorem

In this section we will present a *geometric proof* of the famous theorem of Pythagoras.

Given a right angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides.



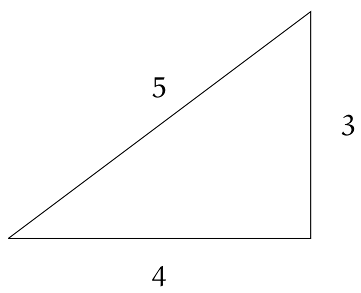
Pythagoras' Theorem:

$$a^2 + b^2 = c^2$$

How might one go about *proving* this is true? We can verify a few examples:

Verify Pythagoras' theorem in the examples below.

1.



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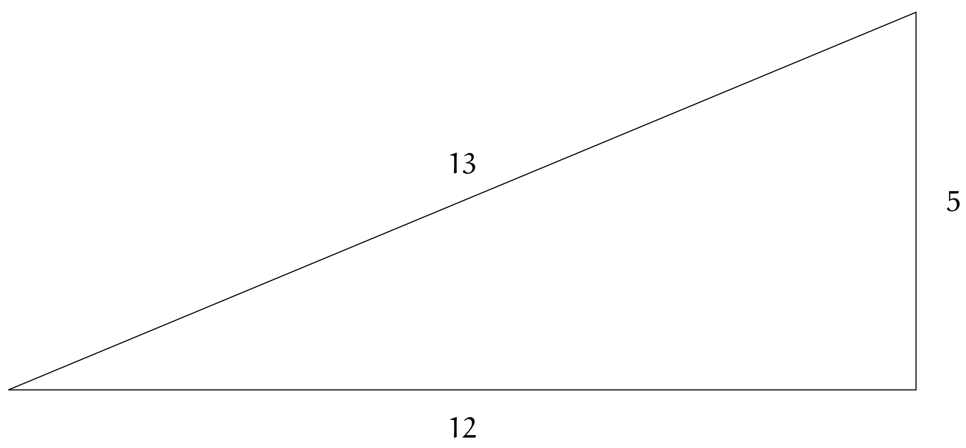
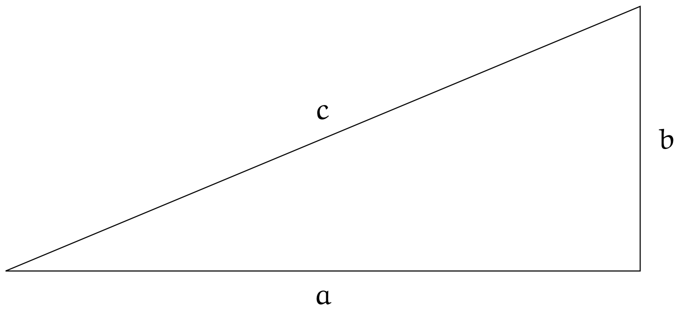
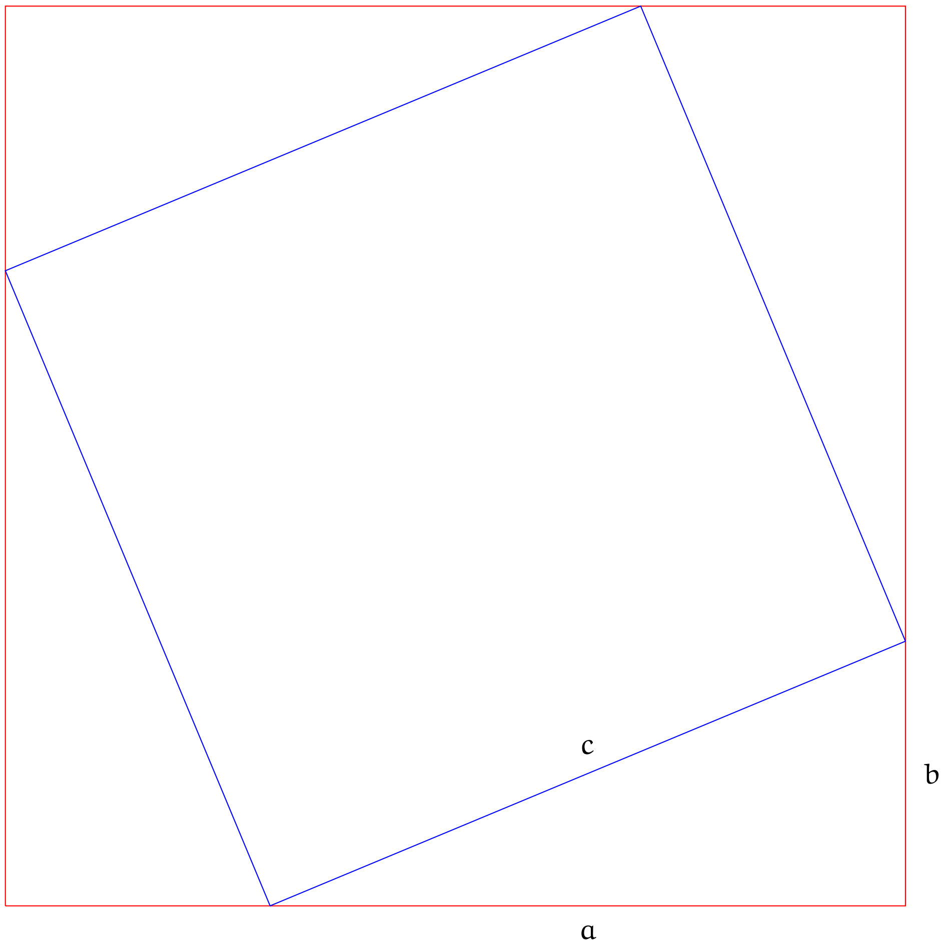


In mathematics this is not considered a proof! Just because this worked in these few examples does not mean that it will *always* work.

We need to give an argument that will work every time. The idea is to use geometry. Start with a general right angled triangle.



We are going to assemble four copies of the right angled triangle into a square, as follows.



1. What is the area of the red square?

2. What is the area of the blue square?

3. What is the area of one of the four triangles?

Now lets put all this together. Verify each of the steps in the following algebraic argument.

$$\text{area of red square} = \text{area of blue square} + (4 \times \text{area of triangle})$$

$$(a+b)^2 = c^2 + 4\left(\frac{1}{2}ab\right)$$

$$a^2 + b^2 + 2ab = c^2 + 2ab$$

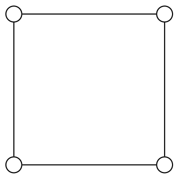
$$a^2 + b^2 = c^2$$

We can see now that the geometric fact about areas translated into Pythagoras' theorem! In this was we can be sure that no matter which right angled triangle you give me, it will ***always*** be true that $a^2 + b^2 = c^2$.

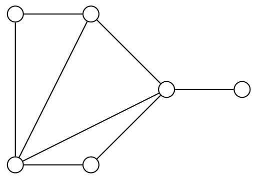
2 Worksheet: Euler Characteristic

A **graph** is a collection of *edges* and *vertices* such that edges are not allowed to pass through one another. Here are some examples:

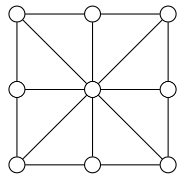
Example 1



Example 2



Example 3

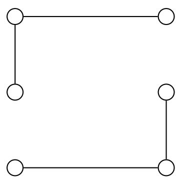


We notice that each of the graphs satisfies the following properties:

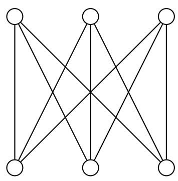
1. Edges only meet at vertices, that is, edges do not cross one another.
2. The graphs are all connected, that is, any vertex can be reached from any other by moving along a path of edges

Here are some bad examples:

Bad Graph 1



Bad Graph 2



Question 1

Draw two examples of graphs that *do* satisfy our rules.

Graph A

Graph B

A **face** of a graph is a region bounded by edges. For instance, Example 1 has 1 face, Example 2 has 3 faces and Example 3 has 8 faces.

Question 2

1. Count how many vertices V, edges E and faces F your two graphs have.

	Graph A	Graph B
V		
E		
F		

2. Now work out the value $V - E + F$ for both of your examples. Compare this value to those of your neighbors. What do you notice?

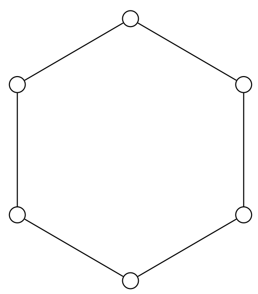
Graph A: $V - E + F =$

Graph B: $V - E + F =$

3. Can you make a guess at what value $V - E + F$ might have in general?

Question 3

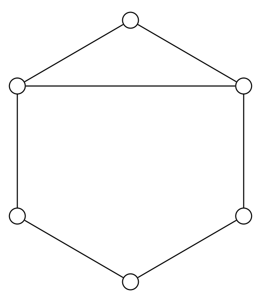
Consider the following graph.



We see that

$$V = 6 \qquad E = 6 \qquad F = 1 \qquad V - E + F = 1$$

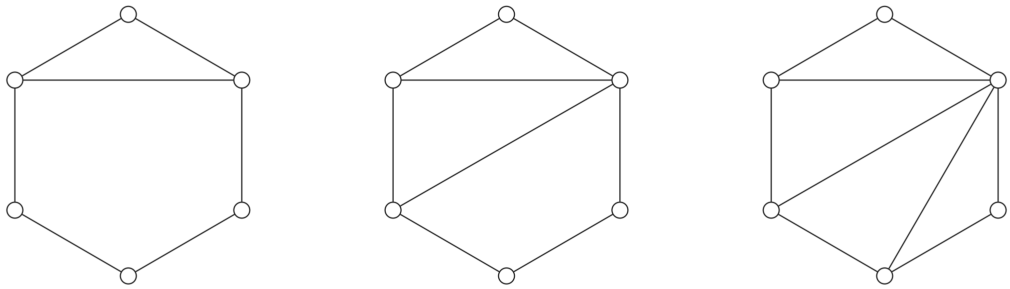
Now we're going to add an edge to this graph



This means that the number of edges went up by 1, and so did the number of faces. That is

$$V = 6 \qquad E = 6 + 1 = 7 \qquad F = 1 + 1 = 2 \qquad V - E + F =$$

Now we repeat this process, adding one more edge and face each time to cut out graph into triangles:

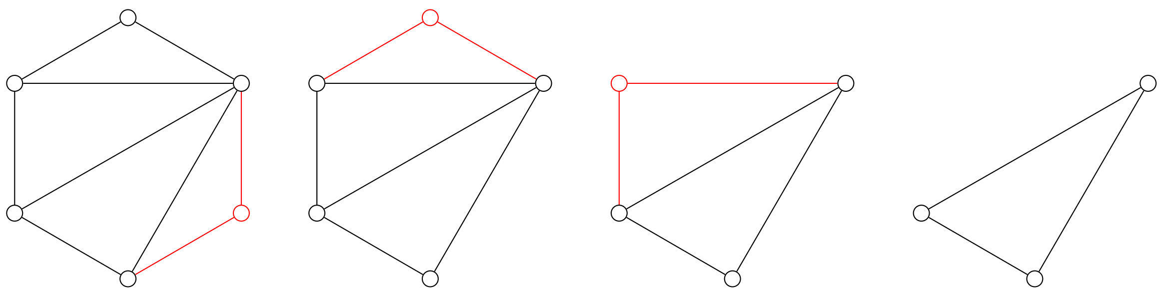


$V - E + F =$ $V - E + F =$ $V - E + F =$

1. What do you notice about the value of $V - E + F$ each time we added a triangle.

2. Can you explain why this happened? [Hint: Think about what we are adding each time, and how that effects each quantity V, E and F.]

3. Now we are going to remove triangles.



$V - E + F =$ $V - E + F =$ $V - E + F =$ $V - E + F =$

Again what do you notice about the value $V - E + F$, and can you explain why this is the case?

3 Worksheet on Induction

This worksheet we will be thinking about a new kind of mathematical proof, called **induction**. As a first example we'll thinking about what happens when we add the first n numbers,

$$1+2+3+4+5+\cdots+n$$

It's often a good idea to begin an investigation such as this by considering a few small examples:

- $n=1$
$$1=1$$
- $n=2$
$$1+2=3$$
- $n=3$
$$1+2+3=6$$

We start to notice a pattern, and make a guess that

$$1+2+3+4+\cdots+n=\frac{n(n+1)}{2}$$

1. Verify this formula holds for the cases $n=1,2$ and 3 above.

$$\begin{aligned} P(1): &\quad 1=\frac{1\cdot 2}{2}=1 \\ P(2): &\quad 1+2= \\ P(3): &\quad 1+2+3= \end{aligned}$$

Does this formula hold for $n=100$? How can we be sure? The answer is mathematical proof, of course! We need to prove that this formula holds *for all* n. We are going to start the induction by proving that $P(4)$ holds without using the formula. Instead we will use $P(3)$, which we have already verified to hold.

$$\begin{aligned} P(4): 1+2+3+4 &= (1+2+3)+4 \\ &= \left(\frac{3\cdot 4}{2}\right)+4 && \text{(use } P(3)\text{)} \\ &= \frac{3\cdot 4}{2}+\frac{2\cdot 4}{2} && \text{(write } 4=\tfrac{2\cdot 4}{2}\text{)} \\ &= \frac{3\cdot 4+2\cdot 4}{2} && \text{(common denominator)} \\ &= \frac{4\cdot 5}{2} \end{aligned}$$

2. Prove that the formula $P(5)$ holds using only the formula $P(4)$. [Hint: follow the same recipe as above in proving $P(4)$]

3. Prove that the formula $P(6)$ holds using only the formula $P(5)$.

Now lets try and do something clever! We notice that we can

- use $P(1)$ to prove $P(2)$
- use $P(2)$ to prove $P(3)$
- use $P(3)$ to prove $P(4)$
- use $P(4)$ to prove $P(5)$

- use $P(5)$ to prove $P(6)$

Can we then use $P(n-1)$ to prove $P(n)$? Let's try and prove that we can!

We assume that we already know the statement $P(n-1)$. In other words this is the statement that the formula holds for a number $n-1$. That formula looks like this.

$$1+2+3+\cdots+(n-1) = \frac{(n-1)((n-1)+1)}{2} = \frac{(n-1)n}{2}$$

What we want to do is *use* this formula to prove the next formula. Verify the following steps in this algebraic argument.

$$\begin{aligned}
1+2+3+\cdots+(n-1)+n &= (1+2+3+\cdots+(n-1))+n \\
&= \left(\frac{(n-1)n}{2}\right)+n \\
&= \left(\frac{n^2-n}{2}\right)+\frac{2n}{2} \\
&= \frac{n^2-n+2n}{2} \\
&= \frac{n^2+2}{2} \\
&= \frac{n(n+1)}{2}
\end{aligned}$$

Notice then that we have just shown that the formula $P(n)$ holds! And all used was the formula $P(n-1)$. Another way to say this is that if we know the formula holds for one number, then we know it holds for the next number. But we can repeat this argument again and again.

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \Rightarrow P(4) \Rightarrow P(5) \Rightarrow P(6) \Rightarrow \cdots$$

We can conclude then that the formula $P(n)$ holds for any natural number n, and this is an example of a proof by induction!

4 On the square root of 2

A history of numbers

The first numbers that people discovered were the counting numbers

$$1, 2, 3, 4, \ldots$$

These are sometimes called the **natural numbers**, because they arise so naturally all around us. For example, you may have 4 apples, and use them to make 1 apple pie, and that might make 6 of your friends very happy.

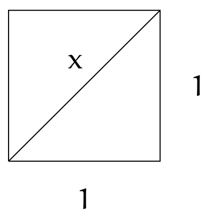
After some time though people realized that there were more numbers lurking around. For example, when we finish the pie there will be 0 pie left (and this makes us sad). So we found the important number *zero*. Soon afterwords negative numbers were hit upon (if we can go from 1 to 0, we should be able to go from 0 to -1, and so on). So we eventually found the whole numbers, which we decided to call the **integers**.

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$$

After a little more pie making occurred, we realised we needed even more numbers! For example, what happens when we are half way through the pie? We needed numbers like $\frac{1}{2}$ and $\frac{3}{4}$. We had discovered the **rational numbers** $\frac{a}{b}$ where a and b are whole numbers, or integers.

The question now is, are we done? Do we have *all* the numbers we could ever need?

Consider the unit square.



How long is the diagonal line? We can answer this question with Pythagoras' theorem.

$$x^2 = 1^2 + 1^2 = 2$$

So x is the number that when squared gives us 2. We denote this number by $\sqrt{2}$ and call it the *square root of* 2. Now is this number one we have already discovered?

1. Square the following numbers:
   (a) $1^2 =$
   (b) $2^2 =$
   (c) $3^2 =$
   (d) $(-1)^2 =$
   (e) $(-2)^2 =$
2. Is $\sqrt{2}$ a natural number?
3. Is $\sqrt{2}$ an integer?

The only possibility we have left is that $\sqrt{2}$ is a rational number. That is, there must exist integers a and b such that

$$\sqrt{2} = \frac{a}{b}$$

Lowest form How many different numbers are there in the list below:

$$\frac{1}{2}, 1, \frac{2}{4}, \frac{5}{5}, \frac{1}{3}$$

We can represent a rational number $\frac{a}{b}$ uniquely by putting it into its *lowest form*, by which we mean, we cannot make either a or b any smaller. Here are some examples:

1. $\frac{4}{16} = \frac{2}{8} = \frac{1}{4}$
2. $\frac{9}{15} = \frac{3}{5}$

Question: Put the following rational numbers into their lowest form.

1. $\frac{2}{6}$
2. $\frac{7}{49}$
3. $\frac{5}{12}$

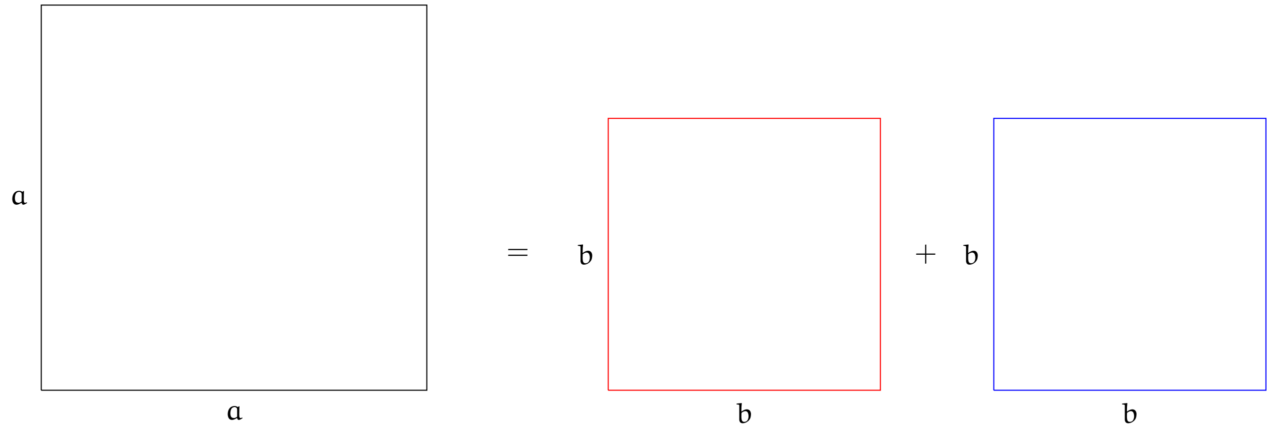
We return to the number $\sqrt{2}$. Suppose that $\sqrt{2}$ is rational. Then we can write it **in its lowest form** as

$$\sqrt{2} = \frac{a}{b} \qquad \text{in lowest form}$$

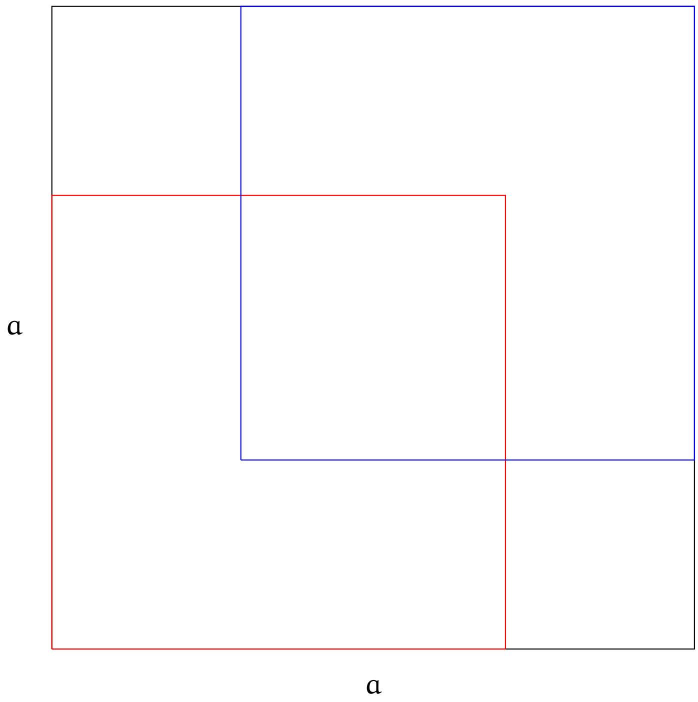
If we multiply both sides by b, and then square both sides, this becomes

$$2b^2 = a^2$$

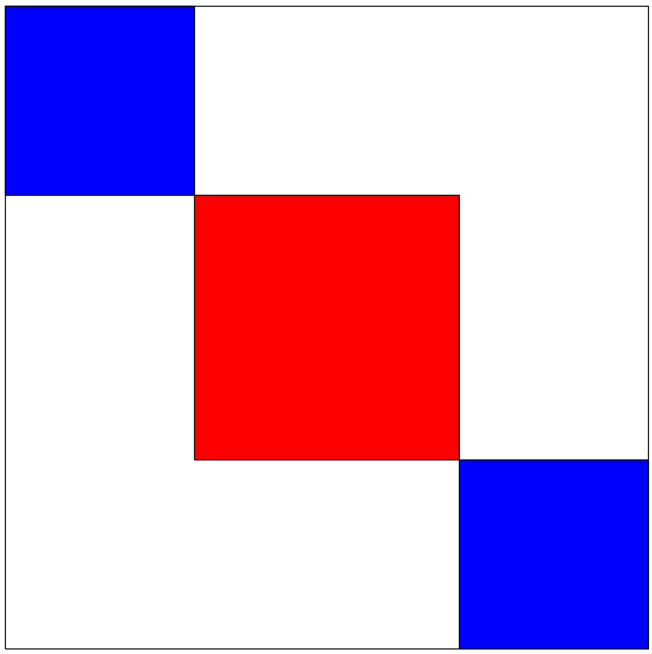
which we can interpret geometrically as follows. The area of the the big square is twice the area of the small square.

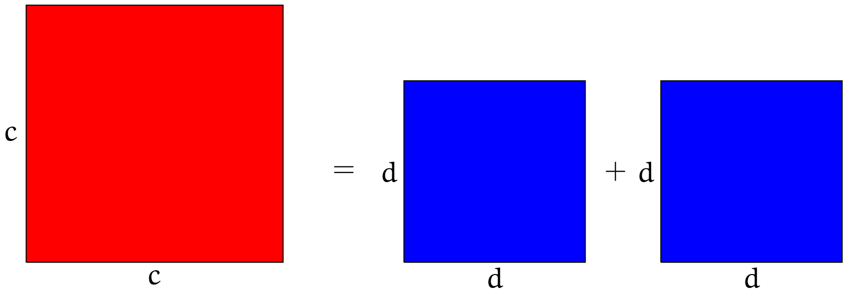


Exercise. Cut out copies of these squares (templates found on the back page). Now place the red square in the lower left corner of the larger square, and the blue square in the upper right corner. Notice that you have covered some part of the large square twice, and some parts not at all.



In particular, notice that since the areas of the two smaller squares is supposed to combine to the area of the larger square, that the part we have counted twice (i.e. the red square below) has the same area as the sum of the two smaller blue squares.





Question:

1. What are the sides lengths c and d of the red square and blue square?
2. Are they integers?

But now we have a problem. Notice that c is smaller than a, and that d is smaller than b. Notice also that we have the following equation.

$$c^2 = 2d^2$$

which if we take square roots and divide by d becomes

$$\frac{c}{d} = \sqrt{2}$$

This means that

$$\frac{a}{b} = \frac{c}{d}$$

but since c and d were smaller this means that $\frac{a}{b}$ was not in it's lowest form! This is a **contradiction**. Therefore we must have made a false assumption. The only assumption we made was that $\sqrt{2}$ was a rational number, so this must in fact be false!

Let's summarize this discussion.

Summary

1. Suppose that $\sqrt{2}$ is a rational number. Then we can write it in its lowest form

$$\sqrt{2} = \frac{a}{b} \qquad \text{in lowest form}$$

2. By considering this geometrically we actually found smaller numbers c, d that were integers and that satisfied the same condition.

$$\sqrt{2} = \frac{c}{d}$$

3. This means that $\frac{a}{b}$ were not in lowest form.
4. This is a contradiction, so our original assumption must have been false! Therefore,

$$\sqrt{2} \neq \frac{a}{b} \qquad \text{not a rational number}$$

We have therefore discovered a new kind of number. This new kind of number is called an **irrational** number, because it is not rational. Another example of an irrational number is the famous number π that relates the radius of circle to its circumference.

Cutting Templates