Prime Examples

Definition: A whole number n is *divisible* by k if n/k is a whole number. We sometimes write $k|n$ to mean that k divides n.
Definition: A whole number p is *prime* if it is larger than 1 and it is only divisible by 1 and itself.

Definition checks!

1. Is 6 divisible by 3? What about 2? What about 4? What about 1? Or 7?

2. Is $3|417$ true? What about $7|417$? Why?

3. How many prime numbers end in a 2? How many end in a 5?

4. How many prime numbers are there? How do you know?

Prime generators

Prime numbers are super important for internet security(Google "RSA security" if you are curious!), so being able to come up with a list of numbers which are prime is super useful. Of the following functions, which do you think return *only* prime numbers for $n \geq 0$? Check $n = 0, 1, 2, 3$ with a calculator if you can. A table of the first sixty prime numbers is provided.

1 - 15	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47
16 - 30	53	59	61	67	71	73	79	83	89	97	101	103	107	109	113
31-45	127	131	137	139	149	151	157	163	167	173	179	181	191	193	197
46-60	199	211	223	227	229	233	239	241	251	257	263	269	271	277	281

1. $2n^2 + 11$

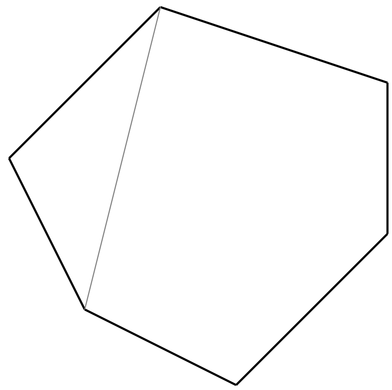
2. $9n + 2$

3. $n^2 - n + 41$

4. $5^n + 2$

5. $2^{2^n} + 1$

Inside-Out



Geometry tells us that the sum of angles in a triangle is always 180 degrees. Is there an equation for the sum of the angles of *any* polygon? You might be able to guess or remember one, but how do you know 100% for certain that it will be true for any polygon?

1. What is the sum of the interior angles of a quadrilateral, a shape with four sides? (Hint: How many triangles are there in a rectangle?)

2. What is the sum of the interior angles of a pentagon? Do you have a guess for the sum of angles in an N-gon, a shape with N sides? Does it work for $N = 3$?

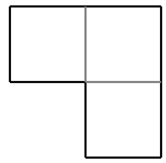
3. Let's say that your guess works for any shape with 7 sides. Does that imply it works for any shape with 8 sides?

4. Let's say your guess works for any shape with n sides. Does that imply it works for any shape with $n + 1$ sides? Does this mean your equation is always true?

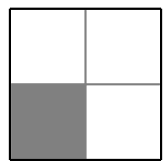
5. Imagine someone is convinced your formula doesn't work for $n = 99999$. How would you convince this person that you are correct, no matter how much they doubt you?

Adventure In Tiles

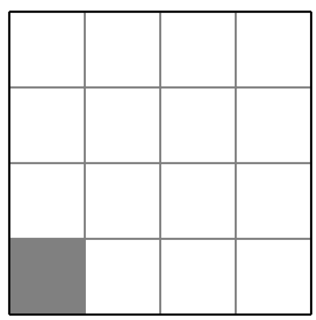
I want to tile my floor. However, my pieces look like this:



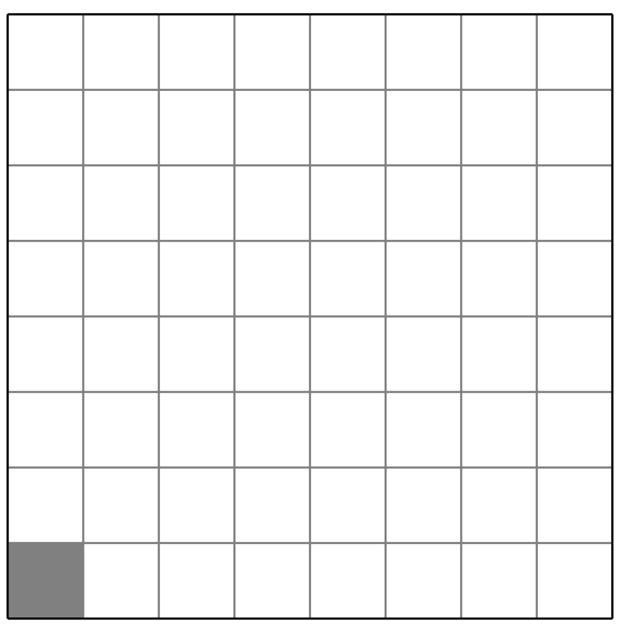
To make things worse, my floor has a weird shape: it used to be 2^n tiles across by 2^n tiles wide, but now I have a beam running through a corner, so one of the corners can no longer take a tile. Can I still completely cover my floor without overlapping any pieces? (Rotating pieces is allowed.)



When $n = 1$, is it possible to – okay actually that's a little silly. But what about $n = 2$? Is a complete tiling possible?

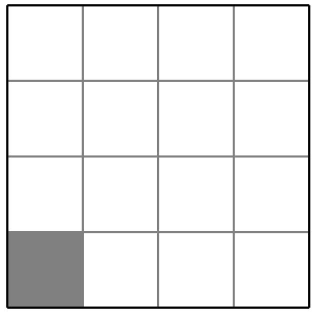


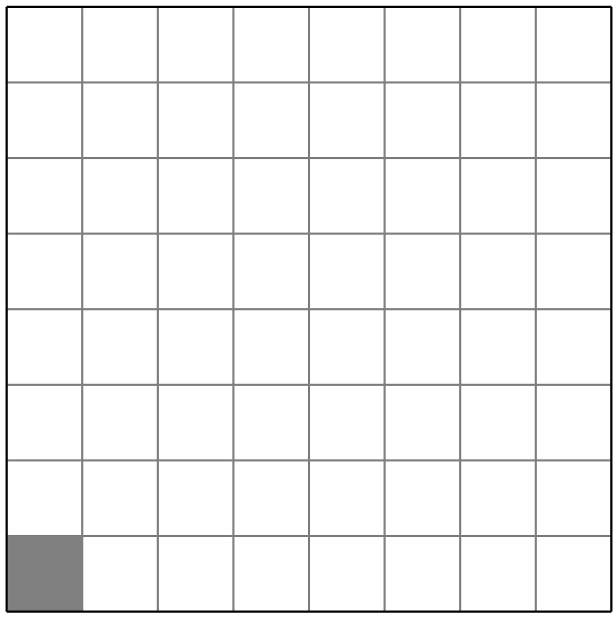
Or $n = 3$?

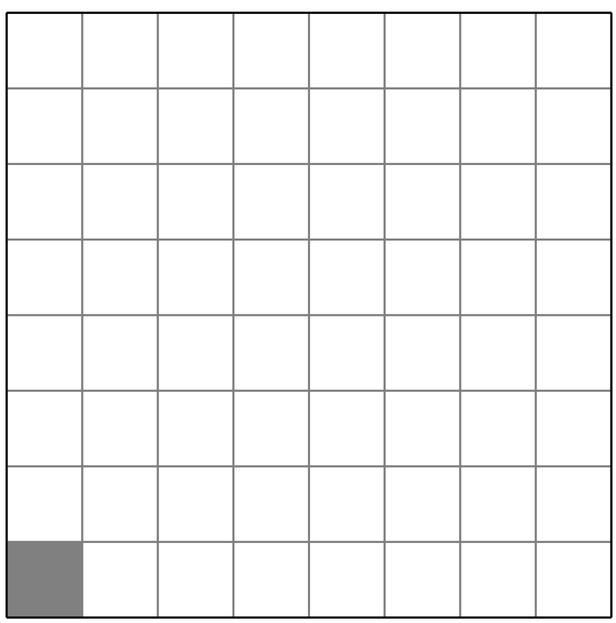


Will it always be possible to tile my floor, no matter how big it is? Why?

(Extra grids)







Bonus : Can you find a formula for how many tiles it takes to cover the n'th floor?

The Amount Crime Pays Is Actually Quite Questionable

Once upon a time, a hundred pirates found a treasure chest with a hundred golden coins. Each pirate had a different idea how to divide the bounty, so they decided to put it up for a vote. As it turns out, pirate democracy looks like this:

- Each pirate has a ~~powerlevel~~ fierceness rank, and each one is at a different rank. Pirate #1, the most fierce of them all, proposes a way to divide up the hundred coins among all 100 pirates.
- Everyone votes on the proposed plan – If it gets a majority or tie, then it passes and the loot is distributed.
- If the plan fails, then pirate #1 gets tossed overboard. Pirate #2, the second most fierce pirate, takes their place, and the process repeats itself.

When voting, each pirate has a ranked list of priorities.

i Firstly, the top priority of every pirate is they want to stay on the boat.

ii The second priority of every pirate is to gain as much gold as possible. If they know they can get even one more coin by tossing someone overboard, they will do it. I mean, they *are* pirates.

iii Their third priority is to avoid tossing people overboard. Unless it will help them stay on the boat or get more money, they won't vote to overthrow the head pirate.

If all the pirates obey these rules and are very clever, how are the hundred coins divided among the hundred different pirates?

Skeleton crews

1. How does the loot get divided if there is only one pirate? What about two pirates?

2. What does this mean when there are three pirates dividing up loot?

The worst solution

1. Assume that the plan from the last case holds whenever there are n pirates voting on loot. Does this mean that the same plan works for $n + 1$ pirates?

2. Using the steps from above, do you have an inductive proof of how the loot must be divided?

Bonus : Salty angry pirates

What if rule (iii) was flipped? That is, a pirate will always vote to overthrow their leader unless they get more money by keeping them. What sort of plan is needed here?

BOUNCE

Back in the day, I lived with a rather lazy twenty-year-old cat called Heisenberg. She had a habit of sitting in place refusing to move for anyone. Everyone loved her, but it was difficult to play with her!

The neglected toy

One day, my friend brought a rubber ball to play with Heisenberg, and they dropped it from 1 meter above the ground next to her (Disclaimer: do not annoy your cats like this). Every time the ball hits the ground it bounces up again, but only reaches half the height from its last bounce.

1. How high does the ball bounce after the first time it hits the ground? What about the second time? The third?

2. Heisenberg will only grab the ball if it bounces a centimeter[1] or less. How many bounces will it take before she finally grabs the ball?

3. If the ball was dropped from a height of h_0, how high will it bounce the first time, h_1?

4. If the ball previously bounced at a height of h_n, how high will it bounce the next time, h_{n+1}?

5. If my friend dropped the ball from a height of h_0, how high will it go on the nth bounce, h_n?

6. How can you be completely sure of your last answer? Will it always work for any $n = 0, 1, 2, ...$?

[1] 1 meter = 100 centimeters.

The lazy toy

Heisenberg still hasn't gotten off the ground, but now she has gotten a bit peeved. Normally, the ball would bounce up to half its previous height, but now Heisenberg has started to swat at it every time it falls. She is a rather precise cat, so the ball now always reaches half a meter above what it normally would.

1. My friend, a little scared by the cat, now drops the ball 1.5 meters above the ground. How high does it go on the first bounce? What about the second time? How is it different than when Heisenberg ignored it?

2. If the ball starts at a height of h_n, how high will it bounce the next time, h_{n+1}?

3. What happens if my friend drops the ball from a height of 1 meter? Does this happen for any other height? How do you know?

4. Let's call h_n the height of the nth bounce of the ball, and define $x_n = h_n - 1$. Rewrite part 2 in terms of x_n and x_{n+1}. What is x_{n+1}? Do you recognize it?

5. If the ball starts at a height of h_0, and $x_0 = h_0 - 1$, what is the value of x_n? How do you know?

6. If my friend drops the ball at a height of h_0, what is the height of the nth bounce h_n? What happens as time goes on, assuming the cat never gets bored?

Indecision Tiles

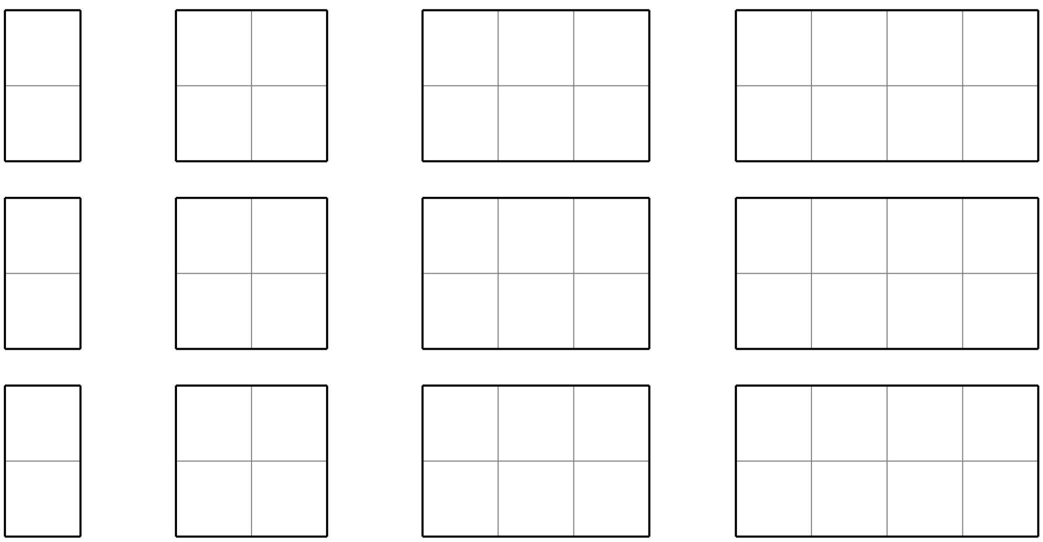
After seeing me tile my home, my friend decided to ask me to help tile their hallway. Because there was a sale, they got tiles that look like this:

Their hallway is two squares high and n squares long, so covering the entire thing isn't too hard. However, there are a lot of different ways to cover the hallway with these tiles, and my friend is chronically indecisive. How many different ways can the hallway be tiled, given that I can rotate the pieces?

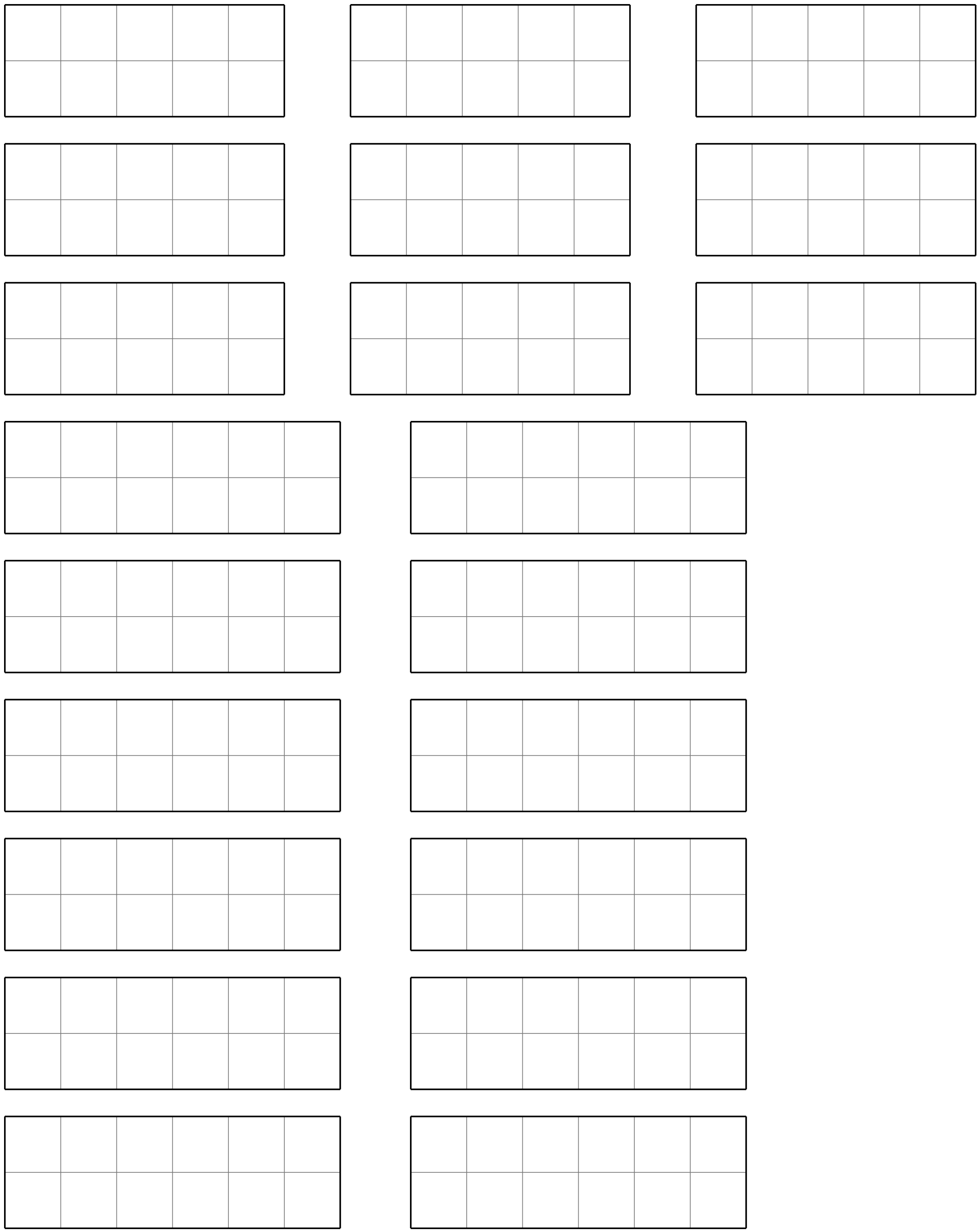
Counting cases

Let's do our usual thing and check the first few cases for n = 1, 2, 3 and 4. How many different ways are there to tile each of these cases?

Let's call f_1 ways to tile in the $n = 1$ case, f_2 the number of ways to tile $n = 2$, and so on and so forth. What do you know about f_n, the number of ways to tile the hallway that's n squares long? Is is possible to find a quick way to calculate f_n for any n?[2]



[2] Hint: think about leftmost tiles.



No Fibbing

The Fibonacci sequence is one of the most popular number sequences out there. It appears not only in math, but also physics, biology, computer science, and is closely related to the ever-popular golden ratio. It is honestly a bit of a cliché how often it shows up!

Definition: The nth *Fibonacci Number* is written as f_n. The first two Fibonacci numbers are defined to be $f_0 = 0$ and $f_1 = 1$, and for any $n \geq 1$,

$$f_{n+1} = f_n + f_{n-1}.$$

Adding up

Using the above, fill in this table with the third to the tenth Fibonacci numbers.

n	0	1	2	3	4	5	6	7	8	9	10
f_n	0	1	1								

Some relations

1. Is it always true that $f_{n+2} + f_n + f_{n-2} = 4f_n$? How do you know?

2. Check that the sum of the first four odd Fibonacci numbers is the fifth even Fibonacci number (that is, check $f_1 + f_3 + f_5 + f_7 = f_8$). Is it always true that the sum of the first k odd Fibonacci numbers is the $(k+1)$'st even Fibonacci number? If not, what's a counter-example?

3. Suppose I say that the sum of the first k even Fibonacci numbers is the k'th odd Fibonacci number. If this is true, try to prove it is so. If not, what should I say instead?

Starting without context

When looking things up online, I came across this equation with no context:

$$f_{m+n} = f_m f_n + f_{m-1} f_{n-1}.$$

What steps should I do understand if this is true for any n and m?

Bonus : Basing a basis

Can you write any number as the sum of Fibonacci numbers? What if you could only use each Fibonacci number once in a sum?

A closed form!

The Fibonacci number f_n can actually be written as a function that just depends on n! We can find this equation, but it will take a little bit of insight.

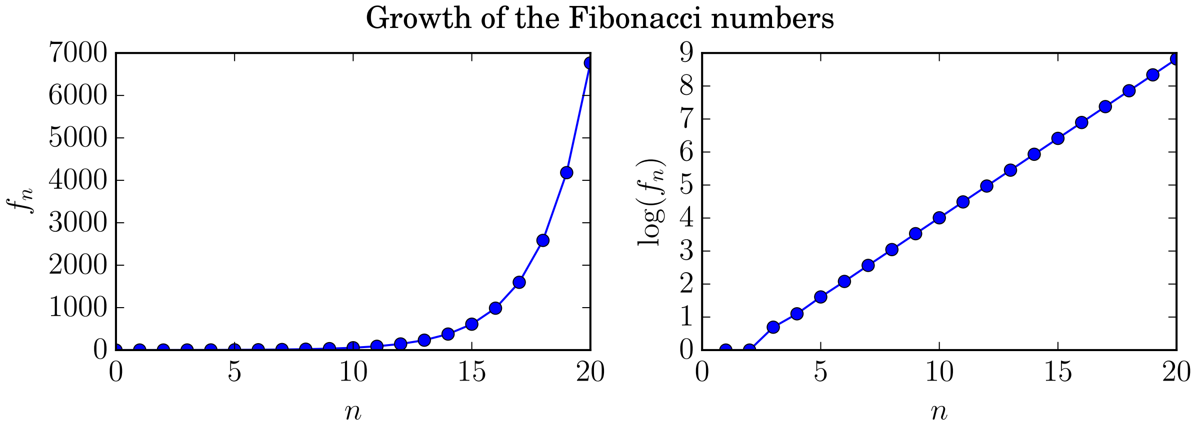
Linearity

Let's say that we have two sequences of numbers, x_n and y_n, and it so happens that both $x_{n+1} = x_n + x_{n-1}$ and $y_{n+1} = y_n + y_{n-1}$ for all $n \geq 1$. Notice: it might be that neither are the Fibonacci sequence, because we don't know the values of x_0, x_1, y_0, or y_1!

1. Given a constant number c, define $v_n = c \cdot x_n$ for all n. What is v_{n+1}? Can it be written in terms of v_n and v_{n+1}?

2. Define $w_n = x_n + y_n$ for all n. What is w_{n+1}? Can it be written in terms of w_n and w_{n+1}?

3. Given constant numbers c and d, define $z_n = cx_n + dy_n$ for all n. What is z_{n+1}? Can it be written in terms of z_n and z_{n-1}? What do you notice? This means that this recursion relation is *Linear*.



Off the deep end (HARD MODE)

To save us some time, I plotted the first twenty Fibonacci numbers. As a Professional Math Human™, I make these sorts of plots whenever I want to guess a good function to fit a recursion.

1. Based on the plots, what kind of function do you expect, ignoring the small values of n? Write it down, and leave a variable wherever you don't know what kind of number you should use.

2. Plug in your guess into $x_{n+1} = x_n + x_{n-1}$. Can you solve for any of the missing variables? Are they all unique? What do you need to do to solve for f_n?

⋆ Proof By Induction ⋆

If you want to prove a statement is always true for any whole number n, then a good way to do that is by using **Proof by Induction**. This uses the following steps:

1. **The Exploration Step:** Either by intuition, trial and error, or computer assistance, you come up with a statement $S(n)$ which you suspect to be true for all n.
2. **The Basis Step:** You establish that the statement works for a few important and fundamental cases. Typically, this just involves checking $S(0)$ for $n = 0$ or $n = 1$.
3. **The Inductive Step:** Assuming that $S(n)$ is true, then use that to show that $S(n+1)$ is true. So $S(n) \implies S(n+1)$.
4. **QED:** The proof is complete!

Both the basis and inductive steps are fundamental to knowing that S is always true – the basis step anchors us to a truth we can trust, and the inductive step lets us climb all the way up to any n we want.

$$S(1) \implies S(2) \implies S(3) \implies S(4) \implies ... \implies S(999) \implies S(1000) \implies S(1001) \implies ...$$

A summing or other

Legends tell of a teacher that was so lazy that they'd tell their students to just add up all the whole numbers from one to a hundred, simply because they wanted to waste time. That stinks.

To stop this, we are going to come up with a formula that we can use anytime, for any n, which equals the sum of all the numbers from 1 to n. But we need to be 100% sure that it will always work, otherwise this teacher has further license to be mean and they certainly don't need more of that. Let's define $A_n = 1 + 2 + 3 + ... + n$ for any whole number n. For example, $A_4 = 1 + 2 + 3 + 4 = 10$.

1. What is A_1? What about A_2? A_3?

2. What is $A_2 - A_1$? What about $A_3 - A_2$? Do you think you know how A_{n+1} relates to A_n? How?

Choose...

Do any of the following equations seem to equal A_n? If you aren't sure, plug in a few values of n and compare.

1. $A_n \stackrel{?}{=} 2n - 1$

2. $A_n \stackrel{?}{=} \frac{1}{2}(n+2)^2 - \frac{3}{2}n$

3. $A_n \stackrel{?}{=} \frac{1}{2}n^2 + \frac{1}{2}$

4. $A_n \stackrel{?}{=} \frac{8}{3}(1.5)^n - 3$

...And commit!

If you've made your choice, then it is time to use induction!

1. **Basis Step:** Does your expression work for $n = 1$?

2. **Inductive Step:** If your expression equals $A_n = 1+2+...+n$ at n, does it equal A_{n+1} when evaluated at $n+1$? (Hint: recall how A_{n+1} is related to A_n).

Bonus : extension

What if I wanted to find $B_n = 1^2 + 2^2 + 3^2 + ... + n^2$ for any n? What steps would you take in order to find an equation for B_n?

Unindicative Induction

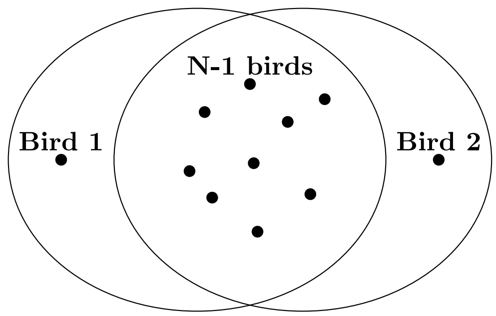
A lot of the time, people write that a proof by induction is just these two steps.

1. We prove a basis case of a statement (i.e., show a statement is true for n = 1).
2. We assume the statement holds in case n, and show it is therefore true in case $n + 1$.

Beautiful birds

Anyway, there is only one kind of bird in the world.

1. **Basis Step:** Any collection that consists of one (1) bird will only have a single color pattern. So the statement holds.
2. **Inductive Step:** We want to show that for any collection of $n+1$ birds, they all have the same color. By induction, assume any collection of n birds has the same species.



Now, whenever we have $N + 1$ birds, we can pick two birds out of the collection and make a picture like the above. So we have bird number 1, bird number 2, and $N - 1$ birds left over.

Look at the left circle: it has exactly N birds in it. Our inductive assumption says that all the birds here have the same color, so we know everyone in this circle will be the same color.

But, the right circle also has exactly N birds in it! So we can apply the assumption again, and say that all the birds in the right circle have the same color.

If we put these two observations together, we know that bird number 1, bird number 2, and the $N - 1$ birds in the middle all have the same color. Therefore, the collection of $N + 1$ birds must all have the same color.

So, using proof by induction, we now know for any collection of n birds, they are all the same color.

But that's wrong

Yep!

But what is wrong with the above proof? Did I use induction incorrectly? Was either my basis step or inductive step wrong[3]? Was there something special about birds? Is induction itself wrong?

[3]Hint: Does the inductive step make sense for every n?

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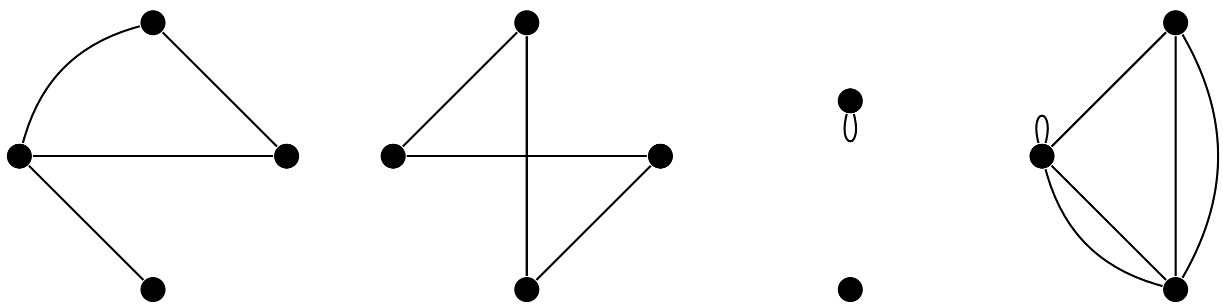
Bonus : Network theory

In math, a **Connected Planar Network** (which is sometimes called a planar graph) is made in the following way

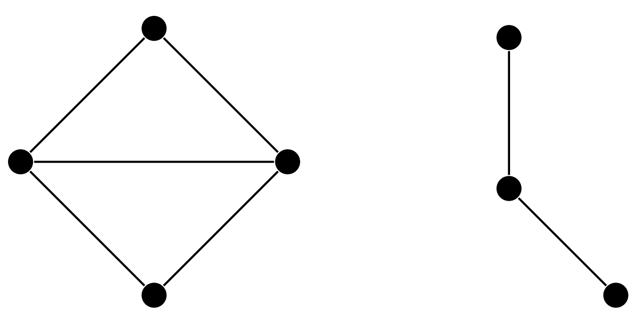
- First, draw some dots anywhere on a page.
- Then, connect the dots together by lines. The lines can curve and bend, but they cannot cross.
- The network must be in one piece – no stray dots or disconnected components.
- A line can't go from a dot to itself, and a pair of dots can't be connected by two separate lines.

GOOD | NOT GOOD | NOT GOOD | NOT GOOD



Let's assign letters to different properties of the network. For example, D = number dots in the network, L = number of lines in the network. A network has a *face* whenever the lines bound an area, and let's say F = number of faces in the network. For example, the planar network above has L = 4, D = 4, and F = 1 (the one face being the triangle). A few more examples of planar networks, with values of L, D, and F, are given below.

D = 4, L = 5, F = 2 | D = 3, L = 2, F = 0 | D = 5, L = 5, F = 1



What does D - L + F equal for each of the above? Is this sum the same for every connected planar network? How do you know?

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A Pitiable Pedometer

My friend wants to use a rubber ball to play with a rather grumpy cat called Heisenberg. My friend is the curious sort, so they attached a motion-tracker to it to see how much Heisenberg will play with it. This device keeps track of how far the ball has fallen in total, ignoring upward motion. For example, if the ball falls from a meter high, hits the ground, bounces up by 1/2 a meter, and then falls to the ground again, then the tracker will display a total distance of $1 + 1/2 = 1.5$ meters.

Naturally, Heisenberg completely ignores the ball and falls asleep. Because she is a cat, and that is simply what cats do.

But we can still read the motion-tracker anyway. Define s_n to be the distance the tracker has logged when the ball has hit the ground for the nth time.

Tracking a tracker

1. Let's say that the ball always returns to half its previous height after a bounce. If the ball was dropped at a height of 1 meter, what is s_3?

2. Let's say that the ball always returns to a fraction r of its previous height after a bounce. If the ball was dropped at a height of a, what is s_4?

3. What is $(1 - r)s_4$? Consequently, how can you rewrite s_4 as a fraction with a denominator of $1 - r$?

4. Using the above, can you guess what s_n is going to be? Does your guess work for s_1?

5. Can you prove your guess always works, no matter what a, r, or n are?

Not your Grecian geometry

Definition: A *Finite Geometric Series* refers to any sum of numbers that looks like

$$A_N := a + a \cdot r + a \cdot r^2 + ... + a \cdot r^N.$$

1. Do you know what A_N equals in terms of a, r, and N?

2. Using the above, can you think of a way to rewrite the number 0.111? What about 0.3333? (Hint: make each digit into a fraction, and the number into their sum.)

3. Let's say I have the number $A_N = 0.33...3$, where there are N "3"'s after the decimal point. What fraction does this look like? What happens as N gets large?

   Almost-Definition: Roughly speaking, a sequence of numbers C_N is defined to have a *Limit* of C_∞ if the value of $|C_N - C_\infty|$ gets arbitrarily small as N gets large.

4. According to your equations, does the sequence A_N from part 3 have a limit? What would this imply for the number 0.33333.....? Is it what you expected?

5. Does this also work for numbers like 0.5555.... and 0.6666....? What is the relationship between limits and decimal expansions?

6. Does 0.999.... equal anything special?

References

[1] http://www.math.cornell.edu/~mec/2008-2009/ABjorndahl/index.html

[2] https://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/Guy697-712.pdf

[3] http://mathworld.wolfram.com/Prime-GeneratingPolynomial.html

[4] https://en.wikipedia.org/wiki/Fibonacci_number

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[6] P. Zeitz, *The art and craft of problem solving*, 2nd edition. (2007)