Side 1: Cardinality

Note: We will use $\mathbb{N}$ to indicate the set of all natural numbers $\{1, 2, 3, 4, ...\}$. We will use $\mathbb{R}$ to indicate the set of all real numbers, i.e. points on the real number line, i.e. decimal numbers.

Let's make a mathematical definition: The *cardinality* of a set is the size of the set. We use the notation $\text{card}(S) = N$ to mean that N is the size of the set S. For example, $\text{card}(\{1, 2, 3\}) = 3$.

1. What's $\text{card}(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\})$?

2. What's $\text{card}(\{\ !\ ,\ @\ ,\ \#\ ,\ \$\ ,\ \%\ ,\ \hat{}\ ,\ \&\ ,\ *\ ,\ (\ ,\ )\ \})$?

3. What's $\text{card}(\{10, 20, 30, 40, 50\} - \{30, 40\})$?

4. What's $\text{card}(\{\})$?

5. What's $\text{card}(\{1, 2, 3, 4...\})$? (i.e. What's $\text{card}(\mathbb{N})$?)

6. What's $\text{card}(\{2, 4, 6, 8, ...\})$?

7. What's $\text{card}(\{1, 2, 3, 4...\} - \{2, 4, 6, 8, ...\})$?

8. What's the cardinality of the set of points on the real number line? (i.e. What's $\text{card}(\mathbb{R})$?)

9. *Warning: Tricky questions. We will work for the next hour to answer these. Just write down your intuitions.* Is $\text{card}(\{1, 2, 3, 4...\}) = \text{card}(\{2, 4, 6, 8, ...\})$? Is $\text{card}(\mathbb{N}) = \text{card}(\mathbb{R})$?

10. Given two big bags of coins, how can one tell - without counting! - whether the two bags of coins have the same cardinality?

Side 2: Bijections

Let's make another mathematical definition: A *bijection* between two sets is a way of pairing elements of one set with elements of the other set.

For example, a bijection between $\{1, 2, 3\}$ and $\{4, 5, 6\}$ is:

$$1 \leftrightarrow 4, \quad 2 \leftrightarrow 5, \quad 3 \leftrightarrow 6.$$

We could also say that the bijection is given by the rule $x \leftrightarrow x + 3$.

1. Can you give a bijection between the sets in #1 and #2 on the last page?

2. Can you give a bijection between $\{1, 2, 3, 4\}$ and $\{10, 20, 30, 40\}$?

3. Is it possible to give a bijection between $\{1, 2, 3, 4\}$ and $\{5, 6, 7\}$?

4. What's card($\{1, 2, 3, 4\}$)? What's card($\{5, 6, 7\}$)? What's card($\{10, 20, 30, 40\}$)?

5. What do the above results suggest about the relationship between the cardinality of two sets and the existence of a bijection between them?

6. Can you give a bijection between $\{1, 2, 3, 4...\}$ and $\{2, 4, 6, 8, ..\}$?

7. What does this tell you about card($\{1, 2, 3, 4, ...\}$) and card($\{2, 4, 6, 8, ...\}$)?

8. Can you give a bijection between $\{1, 2, 3, 4, ...\}$ and the set of all rational numbers (i.e. all numbers of the form $\frac{p}{q}$) where p and q are natural numbers)?

9. What does this tell you about card($\mathbb{N}$) and card(rational numbers)?

10. Is it possible to give a bijection between $\mathbb{N}$ and $\mathbb{R}$?