More Groups

Now that we have a more general concept of a group, we can ask ourselves if are there more examples of groups than the ones that come from symmetries like the ones we have seen so far? The answer is yes! There are lots of groups everywhere in maths, from algebra to number theory, to analysis etc. We will be interested in certain important examples of groups, as well as some of their properties.

1. Pick one of the polygons. In what way do its symmetries form a group? What is the "identity" element? How can you link the idea of an inverse to our previous observations about symmetries?

2. A group that we see all the time in mathematics is the group of integers (whole numbers, both positive and negative), where the operation is addition. How do we know this is a group? If we instead tried to take multiplication as our operation, do the integers still form a group?

3. We can also have very small groups: for example, if we just take the numbers 1 and -1, where our operation is multiplication of numbers, we get a group. How is this a group?

4. Last time we were studying symmetries of regular polygons (whose sides are all the same length). Now try to think of what happens if we let the side lengths differ. What are the symmetries of an isosceles (but not equilateral!) triangle? How about of a rectangle (that is not a square). How many symmetries do they have? Do they have more or fewer symmetries than the regular polygons? What are their symmetry groups?

5. Let's try to understand the third, and perhaps strangest axiom: associativity. Try to figure out in words what this axiom is saying. Consider a butterfly. What are the symmetries of the butterfly? Show that all the possible combinations of the symmetries satisfy the axiom of associativity.

Extension Consider the integers modulo some number n. Show that this forms a group under addition.

Extension What would you need to do to make a group from the integers when the operation is multiplication?