

Stars and Cyclic Groups

1. Let's start with an example. Look at the recycling symbol below:



- (a) What type of symmetry does the recycling symbol have? How many symmetries does it have?

- (b) Let's call r the rotation around the centre point by 120 degrees. Is this a symmetry of the symbol? Can you get all the symmetries of the symbol just by applying this rotation several times?

(c) Let's call e the "do nothing" symmetry, and write composition as multiplication. For example, if we rotate by 240 degrees, this is the same as rotating by 120 degrees twice, so this rotation by 240 degrees will be written as $r \cdot r = r^2$. Can you write all of the symmetries of the recycle symbol either as the element e or as powers of r ?

(d) What happens if you apply r three times? Can you write down an equation from the group elements you found in the previous question?

(e) Write down the inverses of each symmetry.

More generally, if we have a group whose elements are all just powers (or multiples) of a single element g of our group, then we say that the group is a *cyclic group*.

2. Now we're going to try to understand more about these cyclic groups. Given the circles provided, pick a point to start from. For each number k between 1 and the total number n of points on the circle, draw a line connecting your starting point and the k th point around the circle, in a clockwise direction. Repeat this process until you get back to the point where you started.

(a) In any of the choices of k , did you ever find that you never ended up back at your starting point? Why do you think this is?

(b) For each circle, for what choices of k did you hit every point around the circle? How does this depend on the numbers k and n ?

(c) For a given number of total points on the circle, did two different choices of k give you exactly the same star? Can you see a pattern for these pairs of identical stars?

(d) For some choices of k , your stars only used some of the points on the circle before arriving back at the starting point. For which k did this happen? How is this related to the total number of points n on the circle?

(e) Given a star that only hits a subset of the points around the circle, what happens if you rotate all the points in your star by the same amount until your starting point is at an empty point on your circle? Are the rest of the points of your rotated star also on empty points? How many times do you have to rotate your star until every point on the circle is attached to one of your rotated stars? How is this related to the number of points on your star and the total number of points on the circle?

(f) Can you see how to describe these stars in terms of group theory (in particular cyclic groups)?