MEC Fall 2017: Groups around us
Teaching notes and references

The general topic was loosely inspired by the following Math Explorers' Club modules:

- Groups and geometries:
  http://www.math.cornell.edu/ mec/2008-2009/Victor/main.htm
- Groups of symmetries:
  http://www.math.cornell.edu/ mec/Summer2008/youssef/Groups/intro.html

The aim of this program is to introduce the abstract notion of a group via very concrete examples: namely that of symmetries of objects. Each of the axioms of a group are discussed via examples before being put together to then understand the more general idea of a group.

Activities were split over two sessions of 2.5 hours each.

1 Activity 1: Everyday symmetry

The aim of this activity is to understand what symmetry means in everyday life and to try to find a more rigorous definition of symmetry (in particular, planar symmetry) that encompasses this intuitive understanding.
Useful questions include: after choosing one or two objects in the room: "are these objects symmetric? How are they symmetric? Is one object more symmetric than the other?"
For later activities, the idea of transformations of the space (that preserve the object) is very important.
For the final question of the activity, the idea is to choose MC Escher images that display different forms of symmetry (for example the image of crabs that displays translations and reflections, or the image of lizards that displays rotations and reflections) to lead to the idea of classifying different types of symmetry.

2 Activity 2: Types of symmetries

The aim of this activity is now to compare the symmetries of different objects and to gain a feeling of the "number" and types of symmetries an object has. For this activity, the instructor should choose several (approximately 10) examples of objects with different types of symmetry and ask participants to compare them. Good examples include: butterflies, the recycling symbol, honeycomb, friezes, flowers, starfish, mosque ceilings...

Having pictures of the different examples that the participants can physically move and compare is essential.
One way to summarise the results of this exercise that some participants decided to explore is to lay out all the examples by type, with one axis for "amount of symmetry". This visualisation seemed very useful and a nice way to finish the exercise.

3 Activity 3: Counting symmetries

The aim of this activity is to count symmetries (leading to finding the order of the dihedral groups).
The underlying examples are those of regular polygons, with numbers of edges between 3 and 6. Instructors should print out copies of each of these polygons for the participants to manipulate.
Finding ways of not overcounting or undercounting, such as labelling edges or vertices, might not come naturally to participants, and should probably be prompted by the instructor after some time.
The participants shoudl have some paper or a blackboard to write down each of the symmetries they obtain to make sure they count correctly. It is also useful to prompt participants to find these symmetries in a systematic way (all rotations, then one reflection and all rotations) to obtain the correct number.
The participants seemed to find it quite natural to note the pattern between the number of edges n and the number of symmetries $2n$, but this should be the punchline.

4 Activity 4: Putting the symmetries together

The idea of this exercise is to understand the composition of symmetries: applying one symmetry then another, one obtains another symmetry of the object. Participants choose one of the polygons from the previous exercise to study.
Another important idea is showing that a given symmetry (for example, the reflection across one of the diagonals of a square) can be obtained as a composition of other symmetries.
A very important notion that should be emphasised in this exercise is that of the identity of a group - in the context of symmetry this is the symmetry (transformation) in which the object stays still.

5 Summary and definition of a group

At this point, the instructor should be able to give the definition of a group (depending on the background of the participants, the axiom of associativity may be left out - it does not play any real role in the examples explored here). The insrtuctor should lead a discussion on the examples seen previously - how they fit the definition of a group and what all of the axioms mean in the context of symmetries of an object. This should be a good ending point for the first session, or a good way to start the second.

6 Activity 5: More groups

The aim of this activity is to explore exaples of groups that didn't necessarily arise as symmetries of objects previously seen, or seeing objects with fewer symmetries (rectangles versus squares, for example).
An interesting extension to the question of the group structure of the integers that some participants noted on their own was to see what elements should be added to the integers to make it a group under multiplication.

7 Activity 6: Cyclic groups

The aim of this activity is to understand cyclic groups by drawing stars - understanding which elements can generate a given cyclic group, an introduction to the idea of a subgroup and some other interesting structures that depend on the total number of points of the star and the choice of generator.
This activity was found in "Using Star Polygons to Understand Cyclic Group Structure" by Sandy Spitzer, where the explanation of the results can be found:
http://archive.bridgesmathart.org/2012/bridges2012-479.pdf
Materials needed for this activity are circles with different numbers of points placed at regular intervals around the circle (a good selection is circles with 5 points, up to circles with 12 points).
This activity can be adapted easily for different backgrounds: for younger participants, noting different patterns between stars with a prime number of points, generators that are relatively prime with the total number of points, etc can essentially be an exercise in number theory: how results depend on numbers. For older participants, noting which generators lead to subgroups, versus which ones generate the entire group is a more sophisticated explanation of the process.

8 Conclusion

This can be a good place to end the program - with a broad description of how these final examples fit in to the broader framework of groups and abstract algebra. It may be useful to give a real life example of this: "how many ways can you flip a mattress so that it wears out evenly? What group does this form?" is one suggestion.