SPECIAL RELATIVITY

Instructor's Notes

Math Explorer's Club
Spring 2017

These are instructor notes for the Spring 2017 Math Explorer's Club at Cornell University.

1 INTRODUCTION

Einstein's theory of special relativity is based on two fundamental rules about the universe, called **postulates**. Einstein's postulates are:

I. **The Principle of Relativity:** the laws of physics are the same in all inertial reference frames.

II. **The Light Postulate:** the speed of light is the same in all inertial reference frames.

From just these two postulates, we can learn a lot. They have profound consequences for the behavior of time and space, not to mention energy and matter. Moreover, a lot of modern technology depends on the physics of these rules. You may have heard that GPS only works because of relativity, and if you're carrying a cell phone, you have a GPS in your pocket!

2 REFERENCE FRAMES

Both of these postulates depend on the idea of reference frames, so let's talk about those for a second.

[Reference Frames activity]

As we saw in this activity, neither Anna nor Billy was wrong. They were simply looking at things from a different frame of reference: Anna's compass pointed to magnetic north, whereas Billy used the north star to align himself. So each map had a different direction for north, which led to the different position of the landmarks.

This is one example of different reference frames. Another example of reference frames is when one object moves relative to another, say a spaceship flying away from Earth. To the people on Earth, the spaceship is flying away. But to the people in the spaceship, Earth is moving away, and they are stationary. (See an animation here).

[Ask the students: What are some other examples of reference frames? What are some non-examples?]

The other word that goes with "reference frame" in both of the postulates is "inertial." This means that your reference frame is not accelerating relative to another. When you're in an airplane that's taking off, you're pressed back into your seat. If you're on an amusement park ride, you get pushed around as the ride moves. Both of these are examples of reference frames that are not inertial, and these are examples of additional forces that don't exist in inertial frames. So what is an inertial reference frame then?

Inertial reference frames are reference frames that are not accelerating.

3 TIME DILATION

Let's take a look at Einstein's second postulate, which says that the speed of light is the same in all inertial reference frames. How fast is the speed of light? **[Turn the lights in the room on and off.]** Is light infinitely fast?

It turns out that the speed of light is not infinite, and it has a definite value. People first realized this in the 18th century when they noticed that stars were not quite in the same positions when the Earth was on opposite sides of its orbit. Nowadays we know that the speed of light is finite. We also know that it is very fast, because it takes about 3 seconds for a radio signal to get to the moon and back, or anywhere from 5 to 15 minutes for a radio signal to talk to the mars rover and hear a response.

The speed of light is approximately $186,000$ miles per hour. We use the letter c to represent the speed of light.

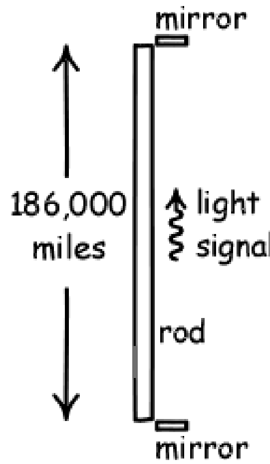
$$c \approx 186,000\ {}^{\text{miles}}/_{\text{hour.}}$$

But wait! This is the speed of light for *every* inertial reference frame. What happens if I'm in one spaceship, and you're in another one that's flying past me. You shoot a laser pointer in the same direction that you're flying. Shouldn't I measure the speed of light to be $186,000\ {}^{\text{mi}}/_{\text{h}}$ plus the speed you're going?

No! That's exactly the point of the second postulate. Both you and I see the speed of light as $186,000\ {}^{\text{mi}}/_{\text{h}}$, no matter the circumstances. This can lead to some weird conclusions, as we'll shortly discover.

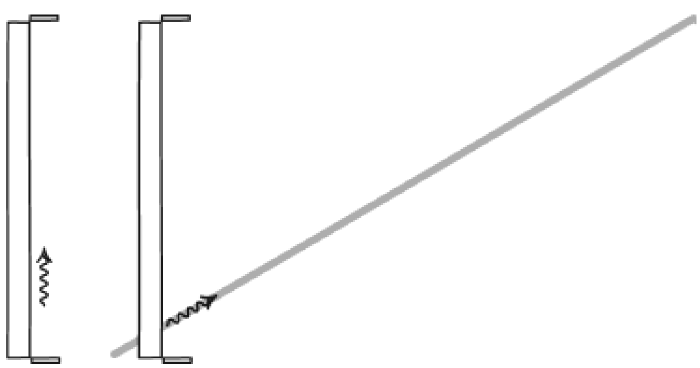
Since the speed of light is the same for everyone, let's use the speed of light to build a super-accurate clock. In fact, this will be the simplest clock possible.

A light clock is an idealized clock that consists of a rod $186,000$ miles long with a mirror at each end. A light signal is reflected back and forth between the mirrors. Each arrival of the light signal at a mirror is a "tick" of the clock. Since light moves at $186,000$ miles per second, the light clock ticks once per second.



(An animation of a light clock ticking is here)

Now imagine that we take this clock and set it moving perpindicular to it's length. Now there's an added complication – when the light signal leaves one end of the rod headed towards the other end, the other end of the rod has moved! So the light signal has to chase the other end of the rod as it flees.



(Also see the animation here)

Here are some questions to consider.

(1) Now that the clock is moving, how does the distance it has to travel compare to the distance it had to travel when it wasn't moving?

(2) Since the speed of light is always the same, does this mean that the moving clock ticks slower or faster than the stationary one?

This means that, if you're standing still, you observe that time passes slower for someone who's moving!

[Clocks activity]

4 A Slower Speed of Light

So now that we've seen some of the strange effects of special relativity with the past activity, you should go experience it for yourselves.

[Students play A Slower Speed of Light]

Once the students have played the game, have them list some of the weird things that they saw. A few possible answers:

- The colors change as you move around.
- Sizes and distances change as you move around.
- It becomes harder and harder to control the character.

5 The Doppler Effect

Probably the biggest change you noticed while playing *A Slower Speed of Light* was that your vision changed when you moved. When you move forwards, you can see into the infrared part of the light spectrum, and when you move backwards, you can see into the ultraviolet part of the spectrum. This is called the **Doppler Effect.** Why does this happen?

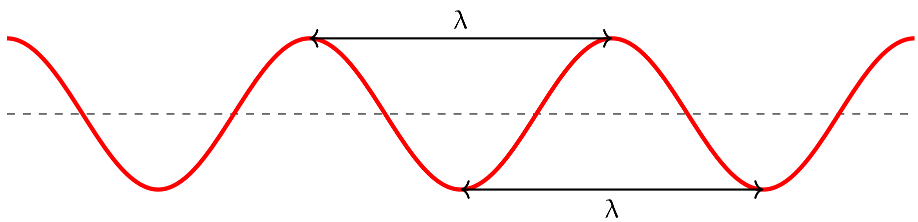
You are familiar with the Doppler effect when it comes to sound. An everyday example of this effect in action comes when hearing the siren of an ambulance or fire engine as it moves past; the frequency (or pitch) of the siren is higher when the ambulance is moving towards the observer, and lower when it is moving away from the observer.

[Demonstrate the doppler effect for the students. Go outside or find a hallway, and have someone run down the hallway while either playing a sound or otherwise making a monotone noise. Notice that the pitch changes!]

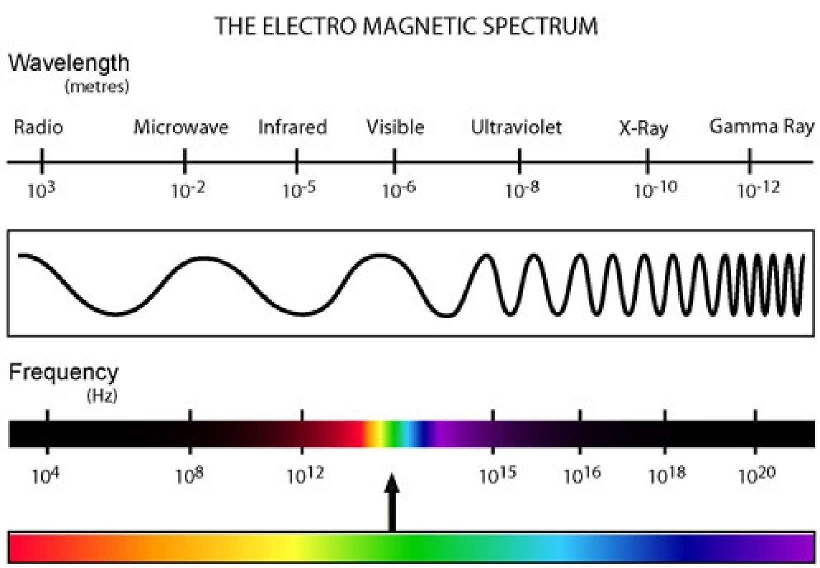
Light travels in a wave, just like sound, so light experiences a Doppler shift too, when you're moving. When you're going near the speed of light, this effect becomes a lot more obvious.

Every wave has three important characteristics: a **wavelength**, a **phase velocity**, and a **frequency**.

The wavelength λ is the space between the peaks of a wave.



Frequency f is how often a wave passes you. For light, higher frequencies correspond to the blue end of the light spectrum, and lower frequencies correspond to the red end of the light spectrum.



The phase velocity v (or just velocity) is the speed at which the wave moves either towards or away from you. These three quantities are connected by the equation

$$v = f \cdot \lambda.$$

When the wave is a light wave, it moves at the speed of light, so the velocity is the speed of light, $c = 300,000 \frac{\text{km}}{\text{s}}$, so the relation between frequency and wavelength becomes

$$c = f \cdot \lambda.$$

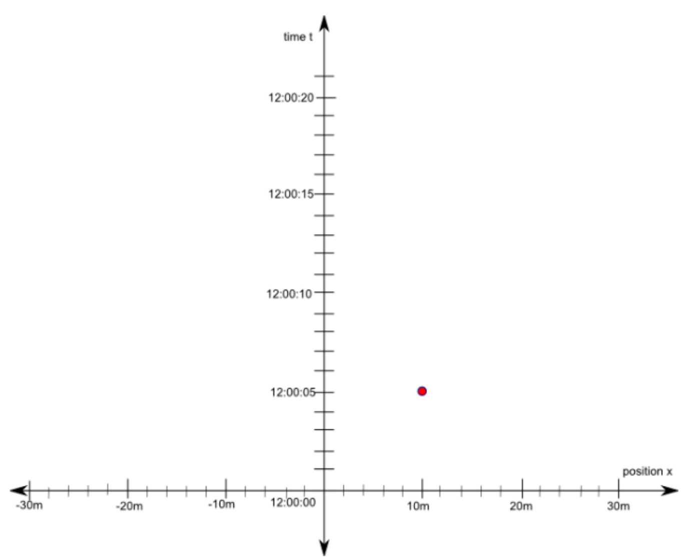
Since the speed of light is a constant, we can figure out the frequency of a wave of light from the wavelength, and conversely we can figure out the wavelength from the frequency.

[Doppler Effect Activity]

6 Worldlines

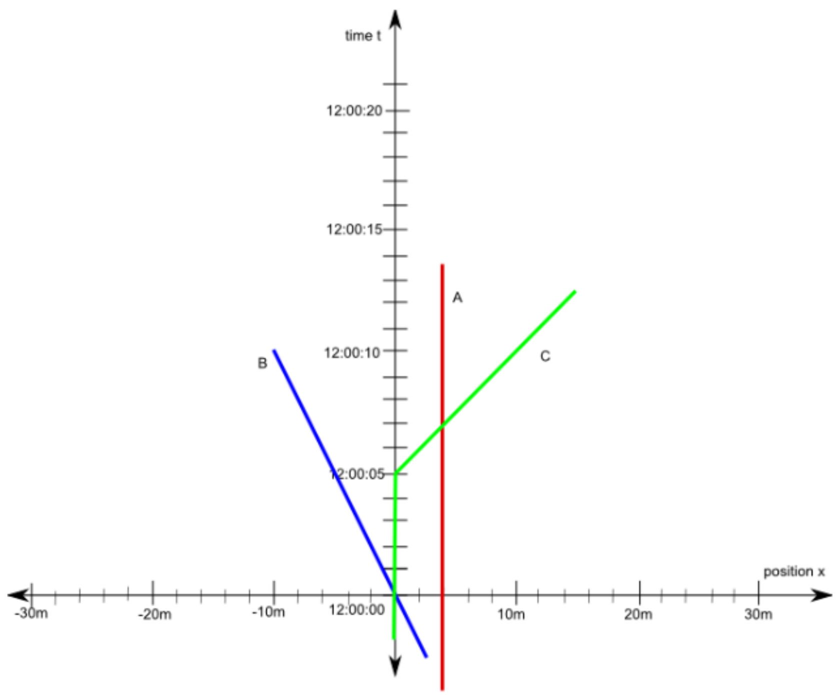
One of the basic philosophical insights in Einstein's theory of relativity is that the three dimensions of physical space and the one dimension of time should not be viewed as completely independent aspects of physical reality, but should instead be viewed as the two components of a unified four-dimensional spacetime on which all other physical objects reside.

Four-dimensional space is not that much more complicated to deal with than three-dimensional space; one simply has to replace three-dimensional vectors (x, y, z) by four-dimensional ones (x, y, z, t). We will focus on a simplified model of spacetime, in which we only consider one dimension of space, together with one dimension of time, leading to a spacetime that is only two dimensional. This allows for spacetime to be displayed on a page or screen in what is known as a **spacetime diagram**. It looks very similar to the familiar Cartesian plots used to graph functions, except that instead of having x and y axes, we have an x-axis (representing space) and a t-axis (representing time).



The points on a diagram of spacetime do not represent points in space, but rather **events** in spacetime. For example, on the diagram above, the dot represents an event 10 meters to the right of a fixed origin, at 5 seconds past noon.

A physical object (such as a human being) does not occupy just one point in spacetime; instead, it occupies a whole curve in spacetime, known as the **worldline** of that object.



For example, in the spacetime diagram above, the red line describes the worldlines of three people, A, B, and C. One of the advantages of spacetime diagrams is that it's easy to tell when or if two objects pass each other.

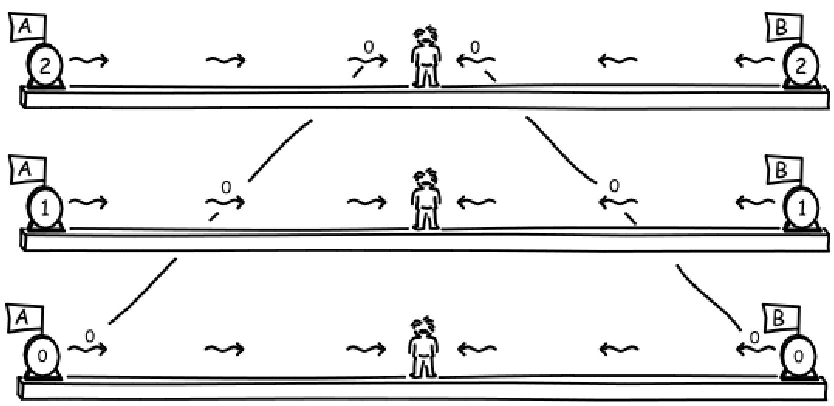
[Worldlines Activity]

7 Relativity of Simultaneity

When Einstein first hit upon special relativity, he thought one effect of special importance, so much so that it fills the first section of his "On the Electrodynamics of Moving Bodies." It is the relativity of simultaneity. According to it, inertial observers in relative motion disagree on the timing of events at different places. If one observer thinks that two events are simultaneous, another might not. At first this will seem like just another of the many novel effects relativity brings. However, as we explore more deeply, you will see that this is the central adjustment Einstein made to our understanding of space and time in special relativity . Once you grasp it, everything else makes sense. (And until you do, nothing quite makes sense!)

There is a quick way to see how this comes about. Imagine a long platform with an observer located at its midpoint. At either end, at the places marked A and B, there are two momentary flashes of light. The light propagates from these events to the observer. Let us imagine that they arrive at the same moment, as they do in the animation below. Noticing that they arrive at the same moment and that they come from places equal distances away, the observer will decide that the two events happened simultaneous.

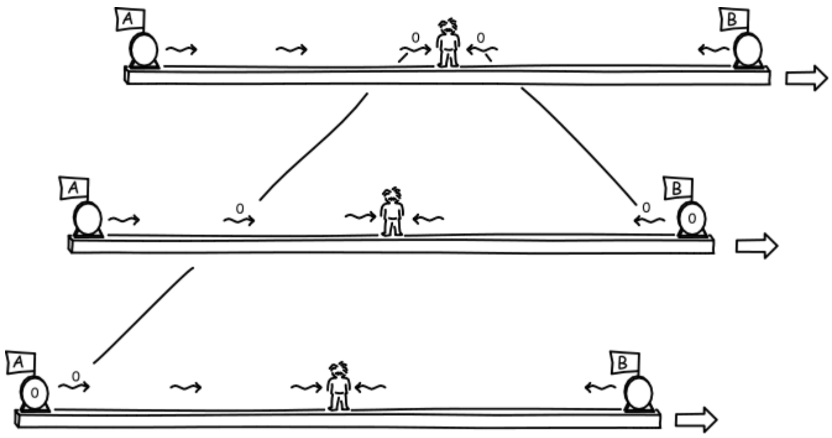
Another outcome is closely related. Imagine also that there are clocks located at A and B. If both clocks show the same reading at the events of the two flashes, then we would judge the two clocks to be properly synchronized. That is what the platform observer judges since, as the animation shows, both clocks read ″0″ when the flashes occur at each location.



See animation here.

So far, nothing remarkable has happened. That is about to change.

Now consider this process from the point of view of an observer who moves relative to the platform along its length. For that new observer, the platform moves rapidly and, in the animation, in the direction from A towards B. Once again there will be two flashes and light from them will propagate towards the observer at the midpoint of the platform. However the midpoint is in motion. It is rushing away from light coming from A; and rushing toward the light coming from B. Nonetheless, the two signals arrive at the midpoint at the same moment.



See animation here.

What is the new observer to make of this? For the new observer, the light from A must cover a greater distance to catch up with the receding midpoint; and the light from B must cover a lesser distance to arrive at the midpoint rushing towards it. So if the two arrive at the same moment, the light from A must have left earlier than the light from B to give it greater time to cover the greater distance to get to the midpoint. That is, the flash at A happened earlier than the flash at B. The two events were not simultaneous, according to the new observer.

The reasoning extends to the clocks. The clocks at A and B show the same time "0" when the flash events happen at each. These two events are not simultaneous for the new observer. Therefore the new observer will judge the clocks at A and B not to be properly synchronized. In fact clock A is set ahead of, that is, earlier than clock B, for it reads "0" earlier than clock B.

This effect is called **the relativity of simultaneity**.

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