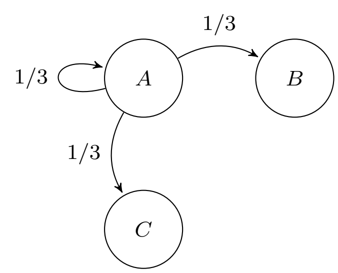
Worksheet 1

Example 1. Imagine you are a frog standing on lily pad A, as in the figure below. There are two other lily pads and you hop to one *randomly*. You are equally likely to jump to any of the lily pads, including the one you're already standing on. In other words, from lily pad A there is a one-third (1/3) chance of jumping to each lily pad.

We represent this situation with the following diagram:



Definition. Notice that we labeled the lily pads A, B, C. We call the lily pads the *states* and the collection of all lily pads the *state space*.

Definition. The *probability transition function* tells us the probability that the frog hops from one state to a different state. The notation $P(x, y)$ tells us the probability the frog hops from state x to state y.

Right now we know that:

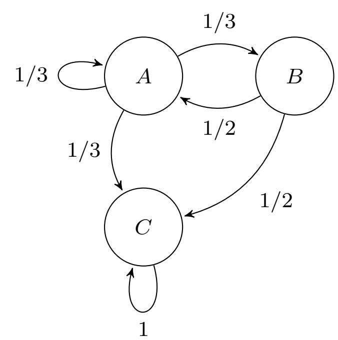
$$P(A, A) = 1/3, \qquad P(A, B) = 1/3, \qquad P(A, C) = 1/3.$$

But we are not done yet! If the frog is on state B or C, where is it going to hop? Let's suppose that we also know the following:

$$P(B, A) = 1/2, \qquad P(B, C) = 1/2 \qquad P(C, C) = 1.$$

Exercise 1. Add arrows to the figure above to account for the new information provided. Are we done yet? Do we need to know anything else about the frog's movement?

The following is the completed state diagram for the frog's jump from page 1:

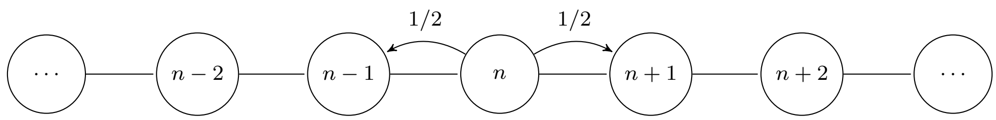


Question 1. What's the sum of the numbers on all the arrows pointing *away from* state A? What about state B? State C?

Question 2. What's the sum of the numbers on all the arrows pointing *toward* state A? What about state B? State C?

Here's a chance to draw some of your own state diagrams and probability transition functions.

Exercise 2 (Random walk in one dimension)**.** Imagine you are standing on the number line. You are equally likely to hop left or right.



What are all the *states* involved? Write out the probability transition function.

Exercise 3 (Random walk in two dimensions)**.** Let's repeat the same idea in two dimensions, on a grid. Now, you are equally likely to hop left, right, up or down. Draw the state diagram that represents a two-dimensional random walk and write the probability transition function.

Exercise 4 (Flipping a coin)**.** What are the states of a coin? Draw the state diagram that represents flipping a coin and write the probability transition function.

Exercise 5.

(a) If a frog starts on lily pad A, what are all the paths it can take in two hops? What is the probability of taking each of these paths?

(b) Add up all these numbers. What do you get?

(c) Using what you just calculated, what's the probability of going from lily pad A to lily pad A in two hops? Going from A to B in two hops? A to C in two hops?

Exercise 6. Come up with a guess of where the frog starting from lily pad A will end up after 100 hops.

Exercise 7. Come up with a guess of which side your coin will be after 1000 flips. Is it as clear-cut as in the previous question?

On a standard 8×8 chess board with no pieces on it, place a knight on a square. Each time it moves the knight chooses a square at random from its legal moves with equal probability. How would you compute the stationary distribution for this problem?