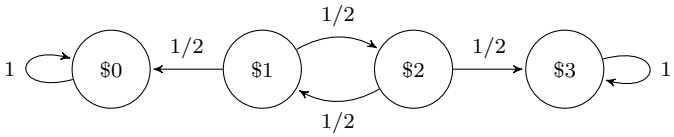
Worksheet 2

Example 1. Consider a frog betting on coin tosses. If the coin lands on heads, it adds one dollar to its fortune; if the coin lands on tails, it loses one dollar. If the frog ever reaches a fortune of 3 dollars, it will stop playing. If the frog runs out of money, then it must stop playing.

Because frogs have bad memory, we provide lily pads labeled $\$0, \$1, \$2, \$3$ to assist the frog. It will hop to the lily pad that corresponds to the amount of money it currently has.

We represent this situation with the following diagram:



Definition. We call the lily pads the *states* and the collection of all labels the *state space*.

Definition. The *probability transition function* tells us the probability that the frog hops from one state to a different state. The notation $P(x, y)$ tells us the probability the frog hops from state x to state y.

Exercise 1. Deduce the value of the probability transition function below from the state diagram above:

$$
\begin{array}{llll}
P(\$0, \$0) = & P(\$0, \$1) = & P(\$0, \$2) = & P(\$0, \$3) = \\
P(\$1, \$0) = & P(\$1, \$1) = & P(\$1, \$2) = & P(\$1, \$3) = \\
P(\$2, \$0) = & P(\$2, \$1) = & P(\$2, \$2) = & P(\$2, \$3) = \\
P(\$3, \$0) = & P(\$3, \$1) = & P(\$3, \$2) = & P(\$3, \$3) =
\end{array}
$$

Exercise 2. Consider another frog that stops betting either when it makes 5 dollars or when it runs out of money. Draw the state diagram for this frog.

Consider the frog from before that bets on coin tosses. Its betting strategy is to bet one dollar at a time until it makes N dollars (win) or until it runs out of money (lose). Let p_k be the probability that the frog wins the game if it starts with k dollars in its purse.

Question 1. Suppose that $N = 2$. What are *all* the ways that the frog can end up winning if it starts with 1 dollar? What is p_1?

Question 2. Suppose that $N = 3$. What are *all* the ways that the frog can end up winning if it starts with 1 dollar? What do these ways have in common?

Question 3. Suppose that $N = 3$. Find the exact value of p_1.

Question 4. Suppose that $N = 3$. Give an equation that relates p_1 and p_2, and use it to find the exact value of p_2.

Question 5. Suppose that $N = 4$. Find equations that relate p_1, p_2, and p_3. Then find their exact values.

Question 6. Can you guess the exact value of p_k for any given N? What equations would you use to show that your guess is correct?

The frog now bets on coin tosses wih the *doubling-up betting strategy*:

- For the first coin toss, it bets one dollar. That is, the frog gains one dollar if it wins this coin toss, and loses one dollar if it loses this coin toss.
- For the second coin toss, it bets two dollars.
- For the third coin toss, it bets four dollars.
- For the fourth coin toss, it bets eight dollars.
- And so on. Basically, the frog keeps doubling its previous bet until it wins; then it stops playing and goes back home. It also goes back home when it doesn't have enough money to play its next bet.

Question 7. Suppose that the frog starts with the fortune of 10 dollars. What is the fortune of the frog if it wins on the first toss? second toss? third toss? fourth toss?

Question 8. Suppose that the frog starts with the fortune of 10 dollars. How would the frog end up losing money in this game? What is the probability that this happens? What if it starts with 20 dollars? with 50 dollars?

Question 9. Suppose that the frog visits the casino to play the same game every day with starting fortune 10 dollars. How much money does the frog make if it wins a game? How much money does the frog lose if it loses a game? How much profit (or loss) would it make on average?

Question 10. Redo the previous problem, but with the starting fortune of 20 dollars. Then with the starting fortune of 50 dollars. What do you notice?

In the Monopoly boardgame, a player moves following the given rules:

1. On the player's turn, the player rolls two dice, sees what number comes up, and then moves that many steps clockwise.
2. If the player lands on the "Go to jail" square, then they must move directly to jail.
3. If the player rolls doubles, then they roll again after completing their turn. A player who rolls three consecutive sets of doubles on one turn has been "caught speeding" and is immediately sent to jail.
4. If a player is in jail, they do not take a normal turn and must either pay a fine of 50 dollars to be released, use a "Get out of jail free" card, or attempt to roll doubles on the dice.

Question 11. Is the game a Markov chain? why or why not? If it is not, what can you change to make the game a Markov chain?

Question 12. How can we use what we have discussed today to help us make a Monopoly strategy?