Averaging for Deterministic and Stochastic Perturbations

Mark Freidlin & Alexander Wentzell.

(1) $\ddot{q}_t = f(q_t)$; $q_0 = q$, $\dot{q}_0 = p$; $p, q \in R^1$.

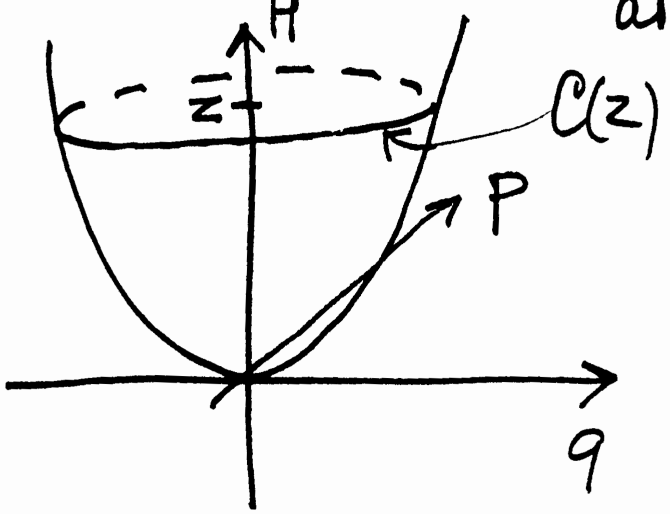
$f(q) = -F'(q)$, $H(p,q) = \frac{p^2}{2} + F(q)$

$$\begin{cases} \dot{p} = -F'(q) = -\frac{\partial H}{\partial q} \\ \dot{q} = p \qquad\quad = \frac{\partial H}{\partial p} \end{cases}$$

$x = (p,q) \in R^2$

$\dot{X} = \bar{\nabla} H(X)$

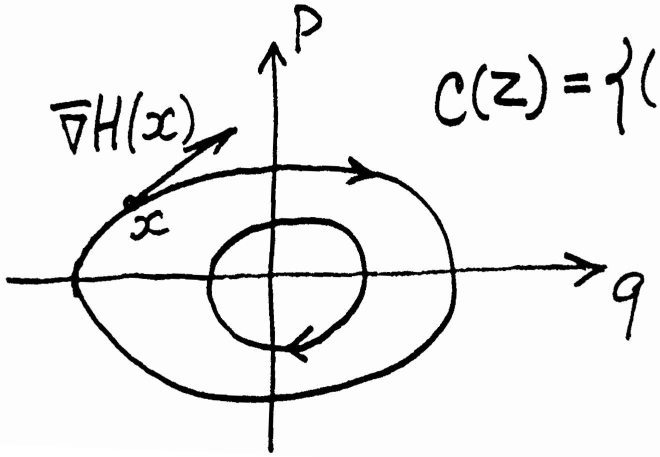
$H(X_t) \stackrel{t}{\equiv} H(X_0)$, The flow X_t preserves the area in R^2.



Invariant density on $C(z)$:

$$m_z(dl) = \frac{1}{T(z)} \frac{dl}{|\nabla H(x)|}$$

$$T(z) = \oint_{C(z)} \frac{dl}{|\nabla H(x)|} \text{ - period}$$



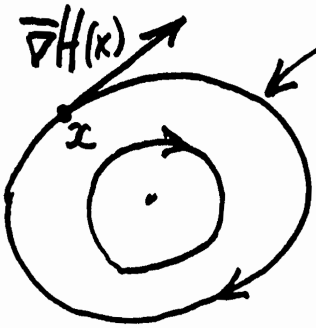
$$C(z) = \{(p,q) : H(p,q) = z\}.$$

② $\dot{\tilde{X}}_t^\varepsilon = \bar{\nabla} H(\tilde{X}_t^\varepsilon) + \varepsilon \beta(\tilde{X}_t^\varepsilon)$

In the case of oscillator: $\ddot{q}_t^\varepsilon = f(q_t^\varepsilon) + \varepsilon \tilde{\beta}(\dot{q}_t^\varepsilon, q_t^\varepsilon)$,
$\tilde{\beta} = -\varkappa \dot{q}$ – classical friction.

<u>Fast</u> & <u>Slow</u> components: $X_t^\varepsilon = \tilde{X}_{t/\varepsilon}^\varepsilon$,

$$\dot{X}_t^\varepsilon = \frac{1}{\varepsilon} \bar{\nabla} H(X_t^\varepsilon) + \beta(X_t^\varepsilon).$$



$$C(z) = \{x \in R^2 : H(x) = z\}$$

$$H(X_t^\varepsilon) - H(x) = \frac{1}{\varepsilon} \int_0^t \cancel{\nabla H(X_s^\varepsilon) \cdot \bar{\nabla} H(X_s^\varepsilon)}\, ds + \int_0^t \nabla H(X_s^\varepsilon) \cdot \beta(X_s^\varepsilon)\, ds$$

Averaging with respect to $m_z(d\ell) = \frac{1}{T(z)} \frac{d\ell}{|\nabla H(x)|}$.

$$H(X_t^\varepsilon) = Y_t^\varepsilon \xrightarrow[\varepsilon \downarrow 0]{C_{0T}} Y_t; \quad \dot{Y}_t = \frac{B(Y_t)}{T(Y_t)}$$

$$B(z) = \oint_{C(z)} \frac{\nabla H(x) \cdot \beta(x)}{|\nabla H(x)|} d\ell = \int_{G(z)} \operatorname{div} \beta(x)\, dx$$

Oscillator, $\beta = -\varkappa \dot{q}$: $B(z) = \varkappa \times$ area of $G(z)$.

③ Stochastic perturbations

$$X_t^\varepsilon \sim L^\varepsilon u(x) = \frac{1}{\varepsilon}\overline{\nabla}H(x)\cdot\nabla u + \beta(x)\cdot\nabla u + \frac{\varkappa}{2}\,\mathrm{div}(a(x)\nabla u)$$

Then (one degree of freedom, simple Hamiltonian)

$H(X_t^\varepsilon) = Y_t^\varepsilon$ – process on $I = \{y \geq 0\}$

converges weakly in C_{0T} to Y_t on $[0,\infty) = I$

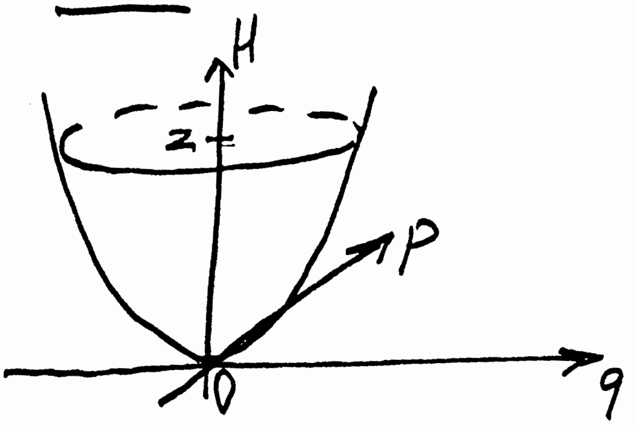
$$Y_t \sim \overline{L}u(z) = \frac{\varkappa}{2T(z)}\frac{d}{dz}\left(A(z)\frac{du}{dz}\right) + \frac{B(z)}{T(z)}\frac{du}{dz},$$

$$T(z) = \oint_{C(z)} \frac{d\ell}{|\nabla H(x)|},\quad B(z) = \int_{G(z)} \mathrm{div}\,\beta(x)\,dx,\quad A(z) = \int_{G(z)} \mathrm{div}(a(x)\nabla H(x))\,dx$$

(Apply Ito's f-la to $H(X_t^\varepsilon)$ and use self-similarity W_t)

Y_t – diffusion process on $I = [0,\infty)$;

Point 0 is <u>inaccessible</u>.



④ Many degrees of freedom, simple first integrals.

$$\dot{X}_k(t) = \overline{\nabla} H_k(X_k(t)), \quad k \in \{1, \dots, n\}.$$

Non-perturbed system: n independent oscillators.
n first integrals: $H_1(x_k), \dots, H_n(x_n)$.
Let each of them is simple, $H_k(x_k) \geq 0$.

Weakly coupled oscillators - perturbed system

$$\dot{X}_k^\varepsilon = \frac{1}{\varepsilon} \overline{\nabla} H_k(X_k) + \beta_k(X_1, \dots, X_n)$$

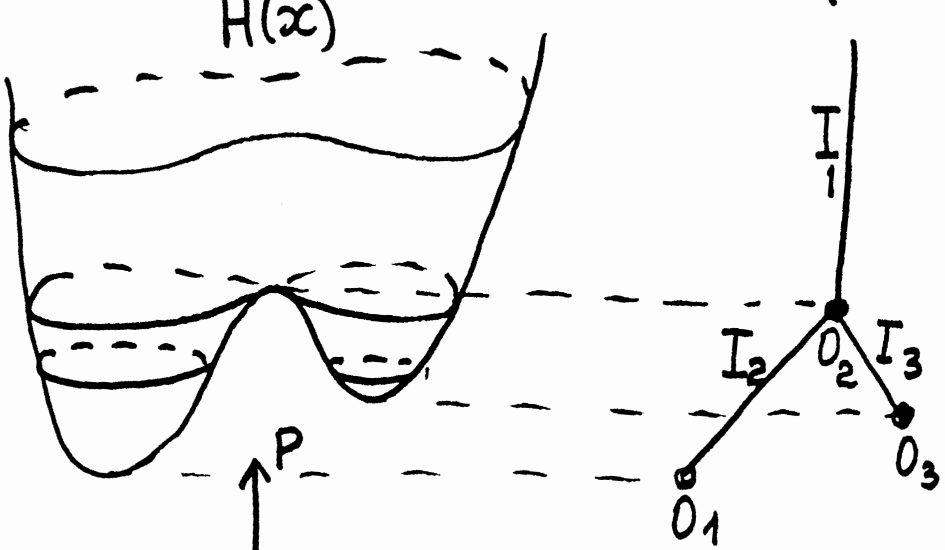
Slow component can be described by
$(H_1(X_1^\varepsilon(t)), \dots, H_n(X_n^\varepsilon(t))) = y_t^\varepsilon \in R_+^n$.

Fast component - motion along non-perturbed trajectories. This is a motion on n-dimensional torus T^n. If the periods $T_1(z_1), \dots, T_n(z_n)$ are rationally independent, the fast motion has the unique invariant density on T^n.

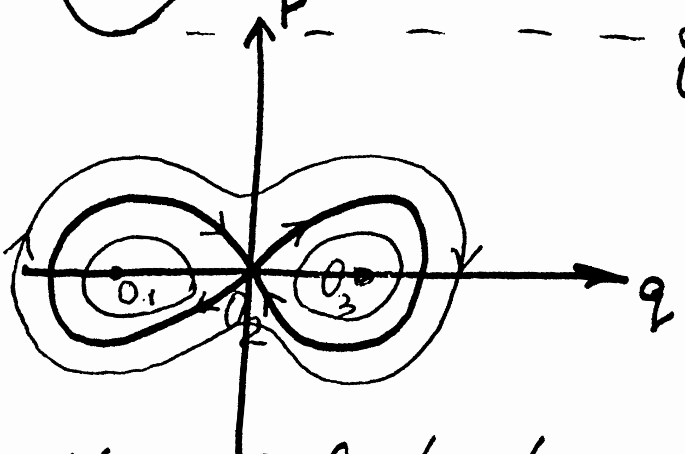
Otherwise, there are infinitely many mutually singular invariant measures.

In the generic situation, rationally dependent tori $T^n = T^n(z_1, \dots, z_n)$ form a dence set in R^n.

(5) Consider again the case of one degree of freedom. But let the Hamiltonian has saddle points.



$$C(z) = \{x \in R^2 : H(x) = z\} = \bigcup_{k=1}^{N(z)} C_k(z)$$



$$\Gamma = \{I_1, \dots, I_n; O_1, \dots, O_m\}$$

$$Y : R^2 \to \Gamma$$

<u>Additional</u> <u>first</u> <u>integral</u>: $k(x)$ = number of the edge in Γ containing $Y(x)$

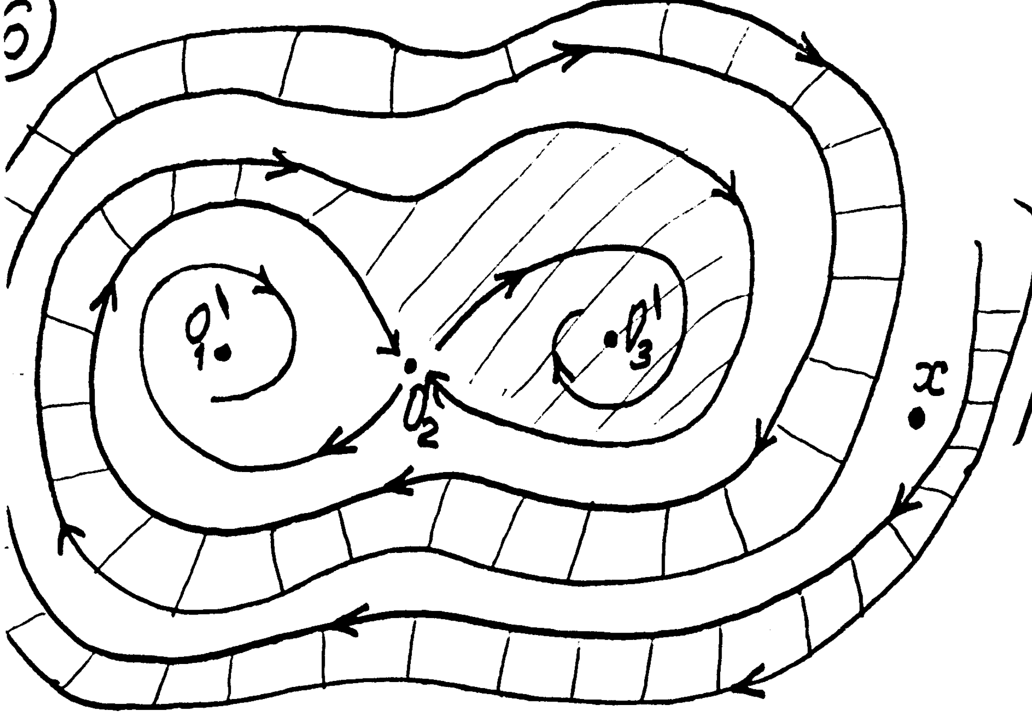
<u>Slow</u> <u>component</u>: $y_t^\varepsilon = Y(X_t^\varepsilon) = (H(X_t^\varepsilon), k(X_t^\varepsilon))$

$$\dot{X}_t^\varepsilon = \frac{1}{\varepsilon} \bar{\nabla} H(X_t^\varepsilon) + \beta(X_t^\varepsilon) + \dots$$

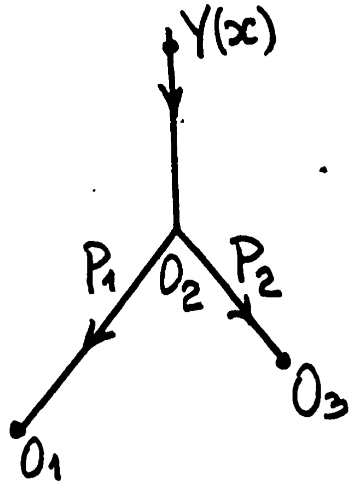
Does the $\lim\limits_{\varepsilon \downarrow 0} H(X_t^\varepsilon)$ exist?

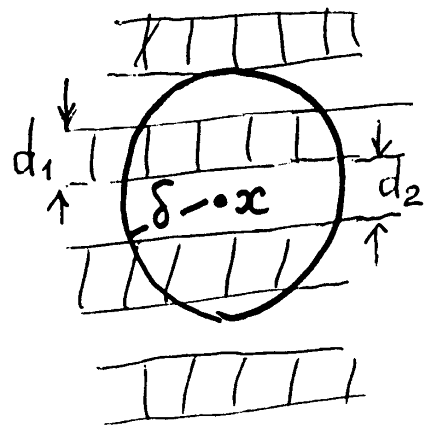
Let $\operatorname{div} \beta(x) < 0$, $H(X_0) > H(O_2)$.

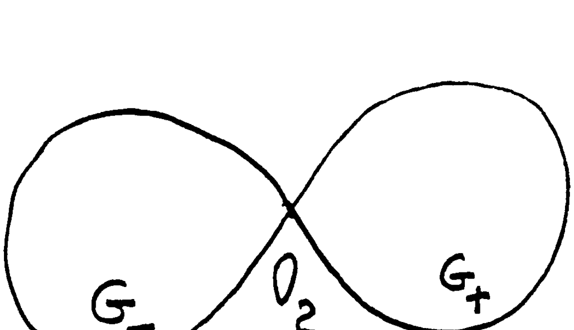
б)



$\lim\limits_{\varepsilon\downarrow 0} H(X_t^{\varepsilon})$ does not exist, if $t > T_0(x)$







$$\lim_{\varepsilon\downarrow 0}\frac{d_1(x,\varepsilon)}{d_2(x,\varepsilon)} = \frac{\int_{G_+}\operatorname{div}\beta(x)\,dx}{\int_{G_-}\operatorname{div}\beta(x)\,dx} = \frac{P_1}{P_2}$$

$\dot{X}_t^{\varepsilon,\delta} = \frac{1}{\varepsilon}\overline{\nabla} H(X_t^{\varepsilon,\delta}) + \beta(X_t^{\varepsilon,\delta})$, $X_0^{\varepsilon,\delta}$ is distributed uniformly in $\mathcal{E}_\delta(x)$.

Then there exist a docle (weak) limit: $\lim\limits_{\delta\downarrow 0}\lim\limits_{\varepsilon\downarrow 0} Y(X_t^{\varepsilon,\delta}) = Y_t$

7. Regularization by random perturbation of the initial point (Anosov, Arnol'd, Neistadt)

$$I=(I_1,\dots,I_n)\in R^n,\quad \varphi=(\varphi_1,\dots,\varphi_m)\in T^m$$

$$\begin{cases}\dot I_t=0\\ \dot\varphi_t=\Omega(I_t)\end{cases}\qquad \begin{cases}\dot{\tilde I}{}^\varepsilon_t=\varepsilon b_1(\tilde I^\varepsilon_t,\tilde\varphi^\varepsilon_t)\\ \dot{\tilde\varphi}{}^\varepsilon_t=\Omega(\tilde I^\varepsilon_t)+\varepsilon b_2(\tilde I^\varepsilon_t,\tilde\varphi^\varepsilon_t)\end{cases}$$

$$(*)\begin{cases}\dot I^\varepsilon_t=b_1(I^\varepsilon_t,\varphi^\varepsilon_t)\\ \dot\varphi^\varepsilon_t=\frac{1}{\varepsilon}\Omega(I^\varepsilon_t)+b_2(I^\varepsilon_t,\varphi^\varepsilon_t)\end{cases}\quad\Bigg|\quad \begin{matrix}I^\varepsilon_0=I_0\in R^n\\ \varphi^\varepsilon_0=\varphi_0\in T^m\end{matrix}$$

Let $(I^{\delta,\varepsilon}_t,\varphi^{\delta,\varepsilon}_t)$ be the solution of $(*)$ with the initial condition distributed uniformly in δ-neighborhood of (I_0,φ_0).

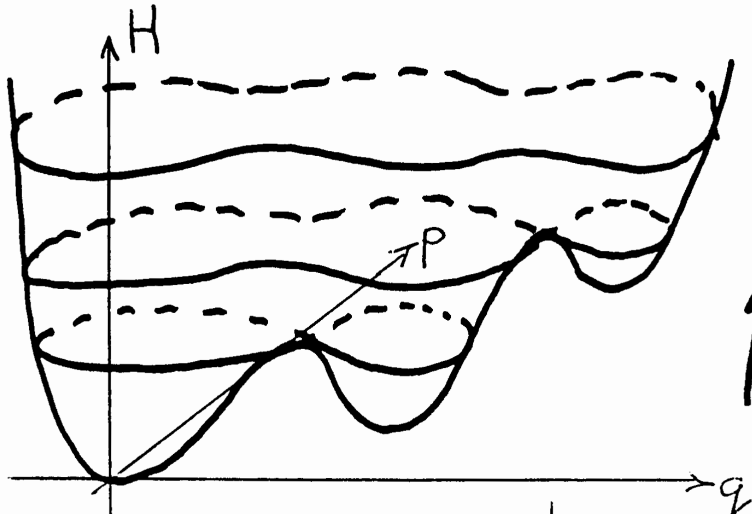
Resonance set: $R=\{z\in R^n: \Omega(z)=(\Omega_1(z),\dots,\Omega_m(z))$ are rationally dependent$\}$.

$$\dot{\bar I}_t=\bar b_1(\bar I_t),\ \bar I_0=I_0,\qquad \bar b_1(I)=\frac{1}{(2\pi)^n}\int_0^{2\pi}\!\!\dots\int_0^{2\pi} b_1(I,\varphi)\,d\varphi.$$

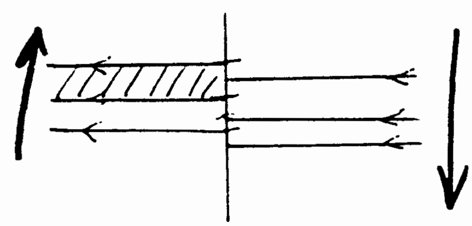
Assume that the set R has Lebesgue measure zero in R^n. Then for any $T>0$, $h>0$,

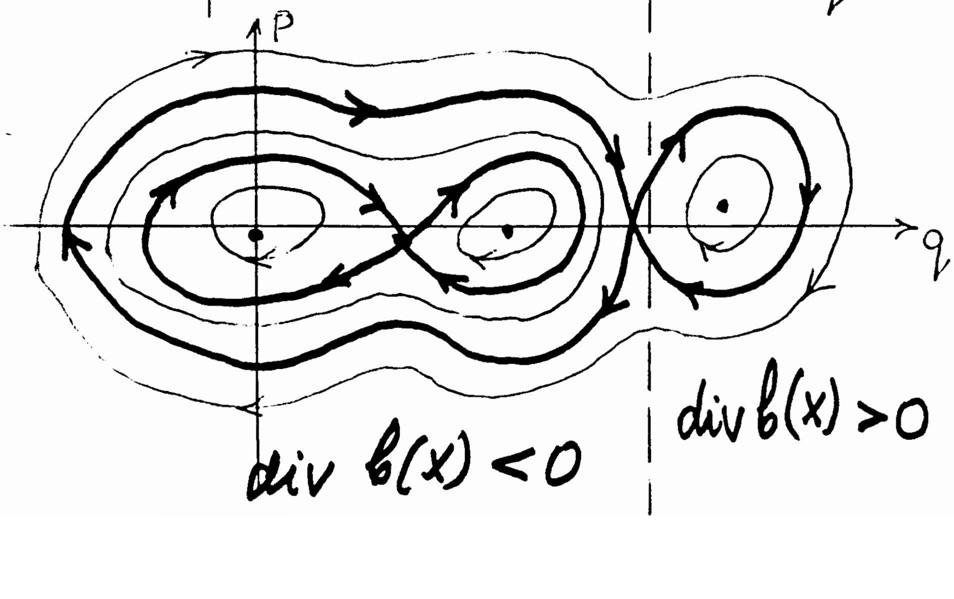
$$\lim_{\delta\downarrow 0}\lim_{\varepsilon}P\{\max_{0\le t\le T}|I^{\delta,\varepsilon}_t-\bar I_t|>h\}=0.$$

(8) The same approach can be used if the system has a saddle point. But if the number of saddle points is greater than 1, stochastic perturbation of the initial condition will not regularize the problem for some (generic) classes of perturbations even in the case of one degree of freedom.



$$\dot{X}_t^{\varepsilon} = \frac{1}{\varepsilon}\overline{\nabla} H(X_t^{\varepsilon}) + b(X_t^{\varepsilon})$$





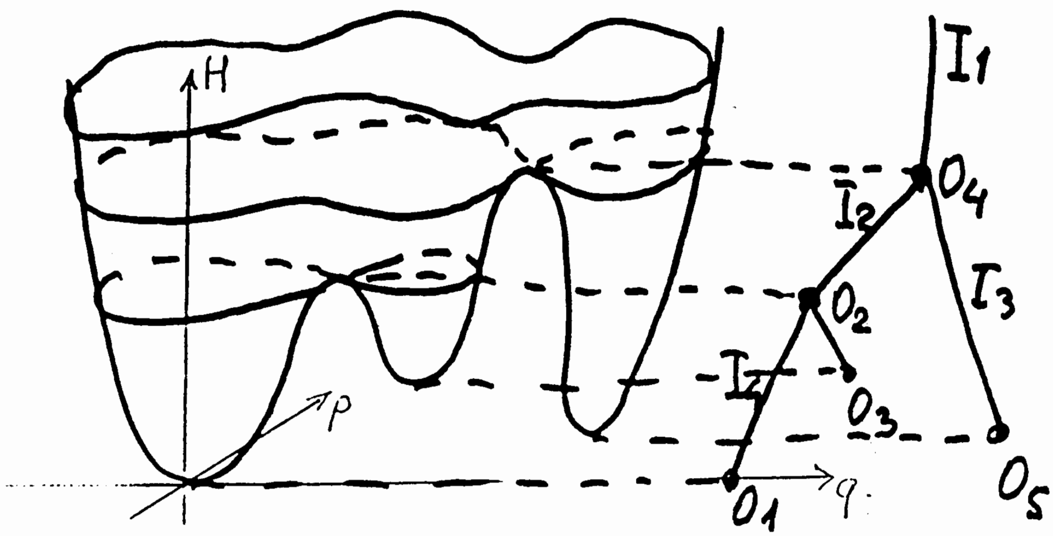
9 Regularization by random perturbation of the equation

$$\dot{X}_t^{\varepsilon,\varkappa} = \frac{1}{\varepsilon}\bar{\nabla} H(X_t^{\varepsilon,\varkappa}) + b(X_t^{\varepsilon,\varkappa}) + \sqrt{\varkappa}\,\sigma(X_t^{\varepsilon,\varkappa})\dot{W}_t$$

$$X_t^{\varepsilon,\varkappa} = x.$$

In the case of one degree of freedom:

$$H(x) \sim \Gamma, \quad Y : R^2 \to \Gamma = \{I_1, \dots, I_n; O_1, \dots, O_m\}$$



<u>stochastic evolution of invariant measures on</u> $C[Y_t]$

<u>Slow motion</u>: $Y(X_t^{\varepsilon,\varkappa}) = Y_t^{\varepsilon,\varkappa} \xrightarrow[\varepsilon\downarrow 0]{\text{Weakly}} Y_t^{\varkappa}$.

$Y_t^{\varkappa}$ - <u>diffusion on Γ.</u>

$$Y_t^{\varkappa} \xrightarrow[\varkappa\downarrow 0]{C_{0T}\text{-weakly}} Y_t$$

Y_t - stochastic process on Γ, the <u>SAME</u> <u>for all σ.</u>
$\det(\sigma\sigma^*) \neq 0.$

(10) $\dot{X}_t = B(X_t)$, $X_0 = x \in R^{n+m}$

Let $Z_1(x), \dots, Z_n(x)$ be the first integrals of the system: $\nabla Z_k(x) \cdot b(x) = 0$.

<u>Perturbed system</u>:

$$\dot{\tilde{X}}_t^\varepsilon = B(\tilde{X}_t^\varepsilon) + \tilde{\delta}^\varepsilon(t, X_t^\varepsilon), \qquad (*)$$

where $\tilde{\delta}^\varepsilon$ is, in some sense, small. Then we can expect that $Z(\tilde{X}_t^\varepsilon)$ changes slowly.

Let, after time change, the perturbed system has the form

$$\dot{X}_t^\varepsilon = \frac{1}{\varepsilon} B(X_t^\varepsilon) + \delta(X_t^\varepsilon) + \sigma(X_t^\varepsilon)\dot{W}_t$$

What phase space Π is good for slow motion

Put $C(z) = \{x \in R^{n+m} : Z_1(x) = z_1, Z_2(x) = Z_2, \dots, Z_n(x) = Z_n\}$

If $C(z)$ consists of one component for each $z \in \Pi$ and the non-perturbed system is ergodic on (most) of these components, then $\Pi = R^n$.

(11) If for some $z=(z_1,\dots,z_n)$ the set $C(z)$ has several connected components, the trajectory of the non-perturbed system remains within the same connected component for all t. This leads to an additional discrete-valued first integral, namely, the number of the connected component within $C(z)$.

Let us identify all points of the same connected component. If the rank of the matrix $(\nabla z_1(x),\dots,\nabla z_n(x))$, $x\in R^{n+m}$, is equal to n for all x, then the space Γ arising after such an identification is an n-dimensional manifold with local coordinates $z_1,\dots,z_n$.

If there are singular points, we have after identification an open book (some number of n-dimensional manifolds (faces) glued together at manifolds of smaller dimension.

12) Weakly coupled oscillators

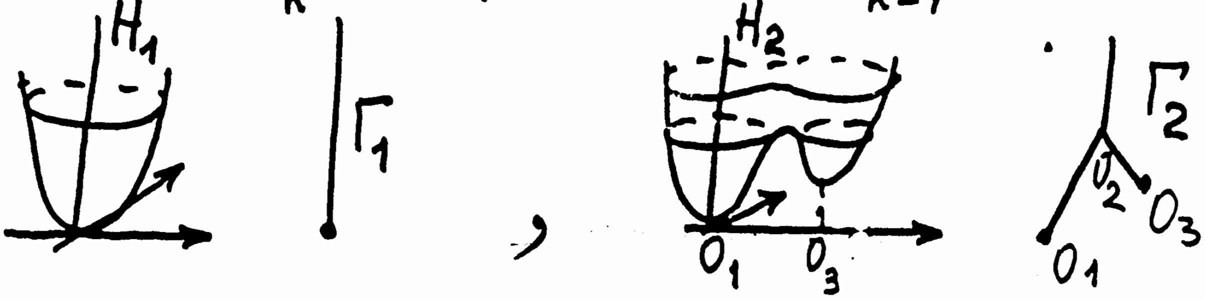
$$\dot{X}_k(t) = \overline{\nabla} H_k(X_k(t)), \quad X_k(0) = x_k \in R^2$$

$$k \in \{1, \dots n\}.$$

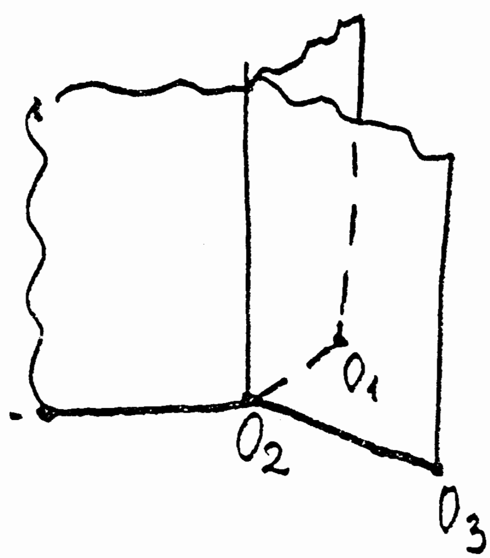
In this case $n = m$.

Conservation laws: $H_1(x_1), H_2(x_2), \dots, H_n(x_n)$

Let $H_k \sim \Gamma_k$, $\Gamma = \prod_{k=1}^{n} \Gamma_k$



$$\Gamma = \Gamma_1 \times \Gamma_2 \quad ; \quad Y: R^{2n} \to \Gamma, \quad Y(x_1, \dots, x_n) = (Y_1(x_1), \dots, Y_n(x_n))$$



$$\dot{X}_k^{\varepsilon}(t) = \frac{1}{\varepsilon} \overline{\nabla} H_k(X_k^{\varepsilon}(t)) + \beta_k(X_1^{\varepsilon}(t), \dots, X_n^{\varepsilon}(t)) + \sigma_k \dot{W}_t^{(k)}$$

Fast component – motion on torus

Slow component: $Y(X_1^{\varepsilon}(t), \dots, X_n^{\varepsilon}(t))$ – process on Γ? Ergodicity.

13) Non perturbed system:

$$\begin{cases} \dot{Z}_i(t)=0, & z(t)=(z_1(t),\dots,z_n(t))\in R^n, \\ \dot{\varphi}_j(t)=\omega_j(Z(t)), & \varphi(t)=(\varphi_1(t),\dots,\varphi_m(t))\in T^m. \end{cases}$$

Perturbed system (after time change):

$$\begin{cases} \dot{Z}^\varepsilon(t) = b(Z^\varepsilon(t),\varphi^\varepsilon(t))+\sigma(Z^\varepsilon(t),\varphi^\varepsilon(t))\dot{W} \\ \dot{\varphi}^\varepsilon(t)=\frac{1}{\varepsilon}\omega(Z^\varepsilon(t))+c(Z^\varepsilon(t),\varphi^\varepsilon(t))+\tau(Z,\varphi^\varepsilon)\dot{W} \end{cases}$$

Put $a(z,\varphi)=\sigma(z,\varphi)\sigma^*(z,\varphi)$, $\bar{a}(z)=\int_{T^m} a(z,\varphi)\,d\varphi$, $\bar{b}(z)=\int_{T^m} b(z,\varphi)\,d\varphi$. Let $\bar{Z}_t$ be the diffusion in R^n with the drift $\bar{b}(z)$ and diffusion $\bar{a}(z)$.

Condition (*): The set of $z\in R^n$, for which the components of the vector $\omega(z)=(\omega_1(z),\dots,\omega_m(z))$ are rationally dependent, has zero Lebesgue measure.

$$\Lambda\left(\left\{z\in R^n:\ \sum_{j=1}^m k_j\omega_j(z)=0 \text{ for some integers } k_j,\ \sum_1^m|k_j|\neq 0\right\}\right)$$

Theorem 1 Suppose $a(z,\varphi)$ is uniformly nondegenerate and condition (*) is satisfied. Then Z_t^ε converges weakly in ... to $\bar{Z}_t$ as $\varepsilon\downarrow 0$.

(14) If we have a completely integrable system

$$\dot{X}_t = \overline{\nabla} H(X_t), \quad X_0 = x \in R^{2n},$$

and $H_1 = H, H_2, \dots, H_n$ are its integrals, then in the area without singularities (under some conditions, Liouville's Theorem), one can introduce action-angle coordinates (I, φ), $I \in R^n$, $\varphi \in T^n$, so that the system has standard form:

$$\begin{cases} \dot{I} = 0, \\ \dot{\varphi} = \Omega(I). \end{cases}$$

Perturbed system:

$$\begin{cases} \dot{\tilde{I}}^{\varepsilon} = \varepsilon b_1(\tilde{I}^{\varepsilon}, \tilde{\varphi}^{\varepsilon}) \\ \dot{\tilde{\varphi}}^{\varepsilon} = \Omega(\tilde{I}^{\varepsilon}) + \varepsilon b_2(\tilde{I}^{\varepsilon}, \tilde{\varphi}^{\varepsilon}) \end{cases}$$

After time change and regularization we have

$$\begin{cases} \dot{I}^{\varepsilon,\varkappa} = b_1(I^{\varepsilon,\varkappa}, \varphi^{\varepsilon,\varkappa}) + \sqrt{\varkappa}\, \sigma_1 \dot{W} \\ \dot{\varphi}^{\varepsilon,\varkappa} = \frac{1}{\varepsilon} \Omega(I^{\varepsilon,\varkappa}) + b_2(I^{\varepsilon,\varkappa}, \varphi^{\varepsilon,\varkappa}) + \sqrt{\varkappa}\, \sigma_2 \dot{W} \end{cases}$$

If condition of Th. 1 are satisfied, we can justify averaging principle: for any initial condition, $\lim_{\varkappa \downarrow 0} \lim_{\varepsilon \downarrow 0} I_t^{\varepsilon,\varkappa} = \bar{I}_t$, $\dot{\bar{I}}_t = \bar{b}_1(\bar{I}_t)$.

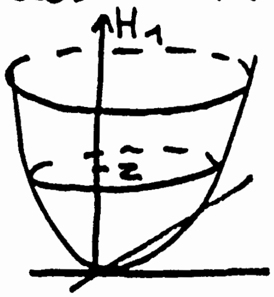
15) Weakly coupled oscillators.

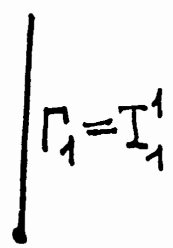
Perturbed system after regularization:

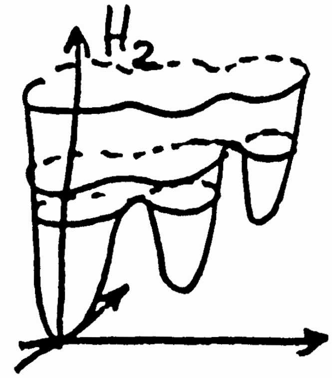
$$\dot{X}_k^{\varepsilon,\varkappa}(t) = \frac{1}{\varepsilon}\overline{\nabla} H_k\left(X_k^{\varepsilon,\varkappa}(t)\right) + b_k\left(X_1^{\varepsilon,\varkappa}(t), \dots, X_n^{\varepsilon,\varkappa}(t)\right) + \sqrt{\varkappa}\,\sigma_k \dot{W}_t^k$$

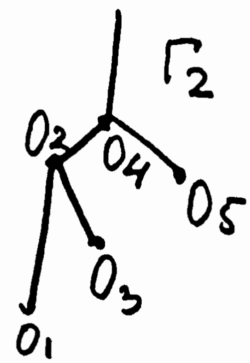
$$H_k \sim \Gamma_k, \quad \Pi = \prod_{k=1}^{n} \Gamma_k, \quad Y: R^{2n} \to \Pi, \quad a_k = \sigma_k \sigma_k^*$$

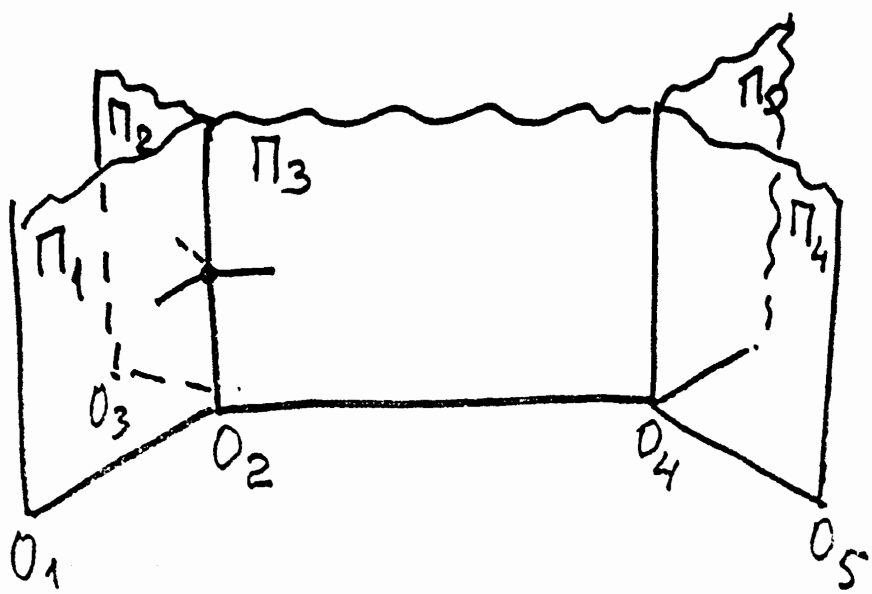
Let $n=2$.











$$\Pi = \bigcup_{i=1}^{5} \Pi_i$$

$$Y(X_t^{\varepsilon,\varkappa}) = Y_t^{\varepsilon,\varkappa} -$$

– process on Π.

(H_1, H_2) – local coordinates on each page Π_i.

Assumptions: (i) $\Lambda\{(z_1, z_2): T_1(z_1) \text{ and } T_2(z_2) \text{ are rationally dep}\} = 0$

(ii) 2×2-matrices a_k are non-degenerate

16) Processes $Y_t^{\varepsilon,\varkappa}$ converge weakly as $\varepsilon \downarrow 0$ to a Markov diffusion process $Y_t^{\varkappa}$ on Π.

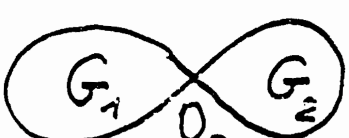
Insid page Π_i, process $Y_t^{\varkappa}$ is governed by the operator

$$\boxed{L_{\varkappa}^{i} = \frac{1}{T_1^i(z_1)T_2^i(z_2)}\left[\sum_{k=1}^{2} \bar{b}_k^i(z_1,z_2)\frac{\partial}{\partial z_k} + \frac{\varkappa}{2}\sum_{k=1}^{2}\frac{\partial}{\partial z_k}\left(\bar{a}_{kk}^i(z_k)\frac{\partial}{\partial z_k}\right)\right]}.$$

$$\bar{b}_k^i(z_1,z_2) = \oint\!\!\oint_{C_1^i(z_1)\times C_2^i(z_2)} \frac{\nabla H_k(x_k)\cdot b_k(x_1,x_2)\,dl_1 dl_2}{|\nabla H_1(x_1)|\,|\nabla H_2(x_2)|},$$

$$a_{kk}^i(z_k) = \int_{G_k^i(z_k)} \operatorname{div}\left(a_k \nabla H_k(x_k)\right) dx_k, \quad T_k^i(z_k)\text{-period } C_k^i(z_k)$$

Gluing conditions on $\{O_2\}\times I_1^1$, $\{O_4\}\times I_1^1$:



$$\boxed{\sum_{\ell=1}^{3} \beta_\ell(O_2)\, D_\ell\, u(y) = 0, \quad y \in \{O_2\}\times I_1^1}.$$

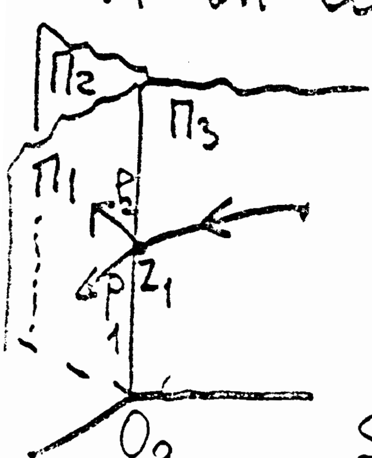
D_ℓ is the operator of differentiation in z_2 in the page Π_ℓ;

$$\beta_i(O_2) = \int_{G_i} \operatorname{div} a_2 \nabla H_2(x_2)\, dx_2, \quad i\in\{1,2\}; \quad \beta_3 = -(\beta_1+\beta_2).$$

(17) The operators $L^i_{\varkappa}$ and the glueing conditions at $\{O_2, O_4\} \times I^1_1$ define process $Y^{\varkappa}_t$ in a unique way; $\{O_1, O_3, O_5\} \times I^1_1$ is inaccessible.

In general, the binding of the book contins parts of co-dimension greater than 1. This part is inaccessible.

Let us now take $\varkappa \downarrow 0$. Then, inside the pages we will have deterministic motion governed by the operators L^i_0. But on the binding of co-dimension one the limit still has stochasticity which depends ONLY on b_k, but not on a_k:



$$P_1(z_1) + P_2(z_1) = 1$$

$$\boxed{\frac{P_1(z_1)}{P_2(z_1)} = \frac{\overline{b}^1_2(z_1, O_2)}{\overline{b}^2_2(z_1, O_2)}}.$$

Stochasticity of the limiting slow motion.