MODELING SEQUENTIAL DISCRIMINATION BETWEEN MARKOV CHAINS

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Abstract (1)

- Appropriateness of the Wald-type logarithmic asymptotics for the mean length (MEAL) of sequential discrimination between composite alternatives.
- Well-known controversy over comparative performances of the asymptotically optimal Chernoff's discrimination strategies and ad hoc heuristic rules of Box and Hunter in the seventies.
- We show a poor performance of the Wald-type asymptotic bounds for the mean length of asymptotically optimal sequential discrimination strategies between simplest types of Markov chains by simulation.
- We propose some partial remedies against this disaster and some alternative asymptotic tools.

Abstract (2)

- **Two case studies**: 1. *First order autoregression with small noise.* Straightforward generalization: sequential discrimination between stable dynamical systems perturbed by small noise.
- 2. Null hypothesis: *Bernoulli binary equally likely sequence* versus all (2×2) transition matrices in certain generalized distance $\geq d^2$ from the null hypothesis, for very small $d > 0$.
- Two *simplified sequential discrimination* strategies between two classes of Markov chains are proposed, and their performance studied by simulation.
- The MEAL of 1. our strategies, 2. Wald-type Lower Bounds, and 3. best static (non-sequential) strategy are compared.

Introduction

$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_0 \cup \mathcal{P}_+$, all disjoint. $P \in \mathcal{P}_r \longrightarrow A(P) : +\mathcal{P}_{1-r}$

Testing $\mathcal{P}_1$ versus $\mathcal{P}_0$, $\mathcal{P}_+$ is *Indifference zone.*

Decision δ at Stopping time N,

α-strategy s: $\sup_{P\in\mathcal{P}} \mathbf{P}_P(\delta \in A(P)) \leq \alpha$.

Wald-type lower bound for MEAL, found for I.I.D., then for MC: $\forall P \in \mathcal{P}$ and $\forall\alpha$-strategies s:

$$\mathbf{E}_P^s N \geq \rho(\alpha, P) + O\left(\sqrt{\rho(\alpha, P)}\right), \tag{1}$$

as $\alpha \longrightarrow 0$, where $\rho(\alpha, P) = |\ln\alpha|/k(P)$

The bound in (1) is asymptotically attained by Wald's Sequential Probability Ratio Test for simple alternatives.

Extended to **controlled discrimination** between composite hypotheses and **change-point detection** problems by Chernoff, Kiefer and Sacks, Lai, etc.

Simulation of comparative performances of the asymptotically optimal Chernoff's discrimination strategies and the ad hoc heuristic rules of Box and Hill (1967) in the seventies (see, e.g. Blot and Meeter, (1973).

Our case studies

1. Illustrative example of testing the correlation significance for a first-order autoregression with small noise.

In certain range of parameters we show the attainability of above asymptotic lower bound *if we permit experimenting in the initial transition period to the equilibrium.*

An extension to the discrimination between conservative dynamical systems perturbed by small noise is straightforward.

2. Two simplified versions of discrimination strategies between Markov Chains (MC) are studied by simulation for testing a regular binary random binary generator versus a MC with *transition probabilities close to those in the null hypothesis.*

Let X be a finite set with m_X elements,

$\mathcal{P}$ be a Borel set of transition probability matrices for MC on state space X. $p(x,y), x \in X, y \in X$, are entries of the matrix $P \in \mathcal{P}$.

Under the convention $0/0 := 1$ we assume that for some $C > 0$

$$\sup_{P,Q \in \mathcal{P}} \max_{x \in X, y \in X} \frac{p(x,y)}{q(x,y)} \leq C < \infty \tag{2}$$

$\forall P \in \mathcal{P}$ MC with transition probability matrix P is **aperiodic and irreducible**, which implies

existence and uniqueness of the stationary distribution $\mu_P > 0 \forall x$.

Log-likelihood probability ratios:

$$z(P, Q, x, y) := \ln p(x, y)/q(x, y).$$

Divergence = cross-entropy

$$I(x, P, Q) := \sum_{y \in X} p(x, y) z(P, Q, x, y).$$

Sets $\mathcal{P}_0$, $\mathcal{P}_1$ and the indifference zone $\mathcal{P}_+ = \mathcal{P} \setminus (\mathcal{P}_1 \cup \mathcal{P}_0)$.

Testing $H_0 : P \in \mathcal{P}_0$ vs. $H_1 : P \in \mathcal{P}_1$, $\forall$ decisions good for $P \in \mathcal{P}_+$.

Suppose that the divergence between the hypotheses is positive, i.e.

$$\min_{i=0,1} \inf_{P \in \mathcal{P}_i, Q \in \mathcal{P}_{1-i}} \max_{x \in X} I(x, P, Q) \geq \delta_0 > 0. \tag{3}$$

The probability for MC $X_i, i = 0, 1, \ldots,$ is denoted by $\mathbf{P}_P$ and the expectation is denoted by $\mathbf{E}_P$. In particular

$$I(x, P, Q) = \mathbf{E}_P(z(P, Q, X_0, X_1)|X_0 = x).$$

A *strategy* s consists of a stopping (Markov) time N and a measurable binary decision δ, $\delta = r$ means that H_r , $r = 0, 1$, is accepted.

Introduce α-*strategies* s satisfying

$$\max_{r=0,1} \sup_{P \in \mathcal{P}_r} \mathbf{P}_P(\delta = 1 - r) \leq \alpha.$$

$\mathbf{E}_P^s N$ is the *mean length* (MEAL) of a strategy s.

Define: $I(\mu, P, Q) := \sum_{x \in X} \mu(x) I(x, P, Q)$,

μ is a probability distribution on X, $I(P, Q) := I(\mu_P, P, Q)$

$I(P, \mathcal{R}) := \inf_{Q \in \mathcal{R}} I(P, Q)$ for $\mathcal{R} \subset \mathcal{P}$; $A(P) := \mathcal{P}_{1-r}$ for $P \in \mathcal{P}_r$ as the alternative set to P (in $\mathcal{P}$).

For $P \in \mathcal{P}_+$, if $I(P, \mathcal{P}_0) \leq I(P, \mathcal{P}_1)$, then $A(P) := \mathcal{P}_1$, otherwise, $A(P) := \mathcal{P}_0$.

Finally, $k(P) = I(P, A(P))$. It follows from (3) that

$$k_0 := \inf_{P \in \mathcal{P}} k(P) > 0, P \in \mathcal{P}$$

It is proved in MT-2001 that $\forall P \in \mathcal{P}, \alpha$-strategies s

$$\mathbf{E}_P^s N \geq \rho(\alpha, P) + O\left(\sqrt{\rho(\alpha, P)}\right), \tag{4}$$

as $\alpha \to 0$, where $\rho(\alpha, P) = |\ln \alpha| / k(P)$.

sketch of α-strategy s^1 attaining equality in (4)

Strategy s^1 consists of conditionally i.i.d. loops.

Every loop contains two phases. Based on the first $N_1(\alpha, \delta_0)$ observations of a loop, we find the MLE $\hat{P} \in \mathcal{P}$.

Enumerate measurements of the second phase anew and introduce $L_k(\hat{P}, Q) = \sum_{i=1}^{k} z(\hat{P}, Q, X_{i-1}, X_i)$.

sketch of α-strategy s^1 (continued)

We stop observations of the loop at the first moment N_2 such that

$$\inf_{Q \in A(\hat{P})} L_{N_2}(\hat{P}, Q) > |\ln \alpha|, or \tag{5}$$

$$N_2 > N_0 := 2k_0^{-1} |\ln \alpha|, \tag{6}$$

stop all experiments and accept the hypothesis H_r (i.e. $\delta = r$), if first of events holds and $1 - r$ is the index of the set $A(\hat{P})$. Otherwise, start a new loop.

We study by simulation simplified strategies s^3 and s^2 comparing (with the prescribed level) only the *likelihood ratios with respect to the closest alternative* to $\hat{P}$ which is numerically much easier to implement.

Note also that for attaining an asymptotic equality in (4) it is assumed in MT-2001 that $\mathbf{P}(N_2 > N_0) \to 0$ as $\alpha \to 0$ making the probability of more than one loop negligible. This holds, if

$$EI(P, \hat{P})/\delta_0 \to 0 \tag{7}$$

as $\alpha \to 0$.

For *close alternatives* (δ_0 *is small)* in both our examples the conventional *asymptotic approach of large deviations* in discrimination problems, see e.g. Chernoff (1972), is inappropriate. Le Cam's theory of *contiguous alternatives might give a better approximation* which we plan to study in future.

The misclassification probability under the hypotheses at distance of order $cn^{-1/2}$, where n is the sample size of the first stage, can be shown to be normal with parameter depending on c.

Choose parameters in such a way that the unfavorable outcome of the first loop would take place with probability less than α.

In sections 3 and 4 the condition (7) is impractical since $L = |\ln \alpha|$ cannot be very large.

Our simulation shows that under the parameters studied only α-strategies with several times larger MEAL than $\rho(\alpha, P)$ seem to be attainable which appears still around twice less than the sample size of the best static strategy.

It is an open problem whether our strategies can be modified to require the MEAL equivalent to the lower bound proved so far.

Remark. The most promising revision of our strategies would be to modify recent methods of **supervised discrimination** *maximizing the margin.* These methods use the estimation of the likelihood function only in the vicinity of the *margin* points crucial for discrimination getting rid of the idea to approximate the likelihood function globally (and plug in the estimated parameters there).

Testing correlation in a first-order autoregression

Consider a Markov chain $X_0, X_1, \ldots$ with joint distribution P_θ:

$$X_t = \theta X_{t-1} + \varepsilon e_t, t = 1, 2, \ldots,$$

where $|\theta| \leq \Theta < 1$ is an unknown correlation, the noise εe_t is $i.i.d.N(0, \varepsilon^2)$ and we **are allowed to choose** X_0 to be, say, 1 (or more generally is random with constant mean as $\varepsilon \to 0$).

We test $H_0 = \{\theta = 0\}$ versus the composite hypothesis $H_1 = \{|\theta| \geq d > 0\}, \{0 < |\theta| < d$ being an indifference zone.

The marginal distribution of X_t is well-known to converge exponentially fast as $t \to \infty$ to $N(0, \varepsilon^2(1-\theta^2))$ for every initial state.

We study here the performance of strategy s^1 for small d, ε and α.

The loglikelihood of P_θ versus $P_{\dot\theta}$ up to moment T is $Z_0 + \sum_1^{T+1} Z_t$, where

$$Z_t := [(X_t - \dot\theta X_{t-1})^2 - (X_t - \theta X_{t-1})^2]/(2\varepsilon^2).$$

First averaging Z_t given X_{t-1}, we get

$$I(x, \theta, \dot{\theta}) = (\theta - \dot{\theta})^2 x^2/(2\varepsilon^2),$$

and then averaging over the stationary distribution, we get the stationary cross-entropy

$$I(\theta, \dot{\theta}) := \mathbf{E}_\theta(Z_t) = (\theta - \dot{\theta})^2(1 - \theta^2).)$$

In particular, $I(0, \theta) = \theta^2, I(\theta, 0) = \theta^2(1 - \theta^2)$.

The Fisher information $J(x, \theta)$ of X_t given that $X_{t-1} = x$ is $x^2/(2\varepsilon^2)$. Thus

$$E(\sum_0^T J(X_t)) = \varepsilon^{-2} \sum_{t=0}^T (\theta^{2t})/2$$

is not less asymptotically for large T than $1/[2(1 - D^2)\varepsilon^2]$ implying that the variance of the preliminary MLE $\hat{\theta}$ based on $\sqrt{\rho(\alpha, P_d)}$ observations is $1/\sqrt{\rho(\alpha, P_d)}$, if we assume that $\varepsilon^2(1 - D^2) = o(d^2/L)$.

This implies the attainment of the lower bound (3) by s^1 along the lines of MT-2001. The bound (4) holds, if the transition period to the stationary distribution is not sufficient to discriminate with error probability less than α, which is also straightforward to rephrase in terms of the model parameters.

Testing Random Number generator vs. MC

H_0 : independent binary equally likely (P_0) sequence $X_1, X_2, \ldots$.

H_1: sequence of stationary MC with transition probabilities $P_r := (p_{ij}, i, j = 1, 2)$, where $r := (r_1, r_2), r_1 := p_{11} - 1/2, r_2 := p_{22} - 1/2$, such that for certain $d > 0$

$$I(P_0, P_r) := -\ln[16 p_{11}(1 - p_{11}) p_{22}(1 - p_{22})]/4 \geq d^2.$$

$I(P_0, P_r) = ||r||_2^2(1 + o(1))$ as $||r||_2 \to 0$, where

$$||r||_2^2 := r_1^2 + r_2^2.$$

For small d approximately the set of alternatives is the exterior to the figure which is close to a circle of radius d with center in $(1/2, 1/2)$.

The Nelder-Mead MATLAB program of finding the set of P_{r^*} minimizing $I(P_{\hat{r}}, P_{\dot{r}})$ over $P_{\dot{r}}$ on the border of the alternative set to a preliminary estimate $P_{\hat{r}}$, always gave a *unique* minimizing point $A(\hat{r})$.

Two simplified algorithms for discrimination between H_0 and H_1

The only difference of the strategy s^2 from s^1 is that we use rule

$$L_{N_2}(P_{\hat{r}}, P_{A_{\hat{r}})} > |\ln \alpha|. \tag{8}$$

The strategy s^3 is greedier than s^2:

we continue to update recurrently the preliminary estimate for the true parameter r during the second phase in parallel to counting the likelihood ratios,

and if a loop ended undecidedly, we plug the updated estimate for r into the likelihhod ratio, find the closest alternative, and start the new second phase.

Static Discrimination, error probability α

Our large sample discrimination problem is equivalent to the discrimination of the zero mean hypothesis for the bivariate rotationally invariant Normal distribution vs. the spherically invariant alternative dealt with in Example 5.17 of Cox and Hinkley (1974):

Best critical region = exterior of a circle of certain radius,

distribution under the alternative is the non-central Chi-Square with df=2.

Power diagrams of non-central Chi-Square imply:

radius providing the equality of maximal errors under the null hypothesis and the alternative is approximately $0.39d$.

Derived sample size is compared with the empirical MEALs of strategies s^3, s^2 by simulation.

Simulation Results

Several series of simulation results for s^2 and s^3

done by MATLAB program (available by request) with various parameters of the model.

In the table below summarizing our simulation ,

$N_1 = K_1\sqrt{L}/d^2$, $L = |\ln(\alpha)|, n$ is the No. of times strategies were repeated,

the empirical MEAL (EMEAL) is the average No. of random numbers taken before the decision, the ENOL is the average No. of loops over n runs.

All parameters in simulations 2-4 are taken the same with n=100, and with $n = 1000$ in simulation 5 for estimating the variance of the performance parameters.

In the plots $d^2 = 0.001, p := p_{11} = p_{22}$, methods 1 and 2 mean respectively s^2 and s^3.

Figures 3 and 6 plot respectively EMEAL and FE under various alternatives.

Other figures illustrate the performance of our strategies under H_0.

EMEAL, number of loops and empirical error rate are plotted versus changing values of various parameters of our strategies.

Main news is that the EMEAL exceeds the theoretical principal term approximately four times under best parameters of our strategies.

Simulation Results (5 single trials)

		d^2	α	K_1	p	n	EMEAL	FE	ENOL
1	s^2	0.0002	0.01	2500	0.5015	200	263723.82	0.02	4.37
	s^3						93836.39	0.0	1.905
2	s^2	0.001	0.02	500	0.5	100	42434.89	0.09	4.27
	s^3						15553.15	0.03	1.89
3	s^2	0.001	0.02	500	0.5	100	49191.76	0.04	4.8
	s^3						15879.13	0.01	1.97
4	s^2	0.001	0.02	500	0.5	100	52232.04	0.04	4.97
	s^3						16412.41	0.03	1.98
5	s^2	0.001	0.02	500	0.5	1000	42934.085	0.057	4.24
	s^3						15729.03	0.01	1.90

Fig. 1: EMEAL versus α

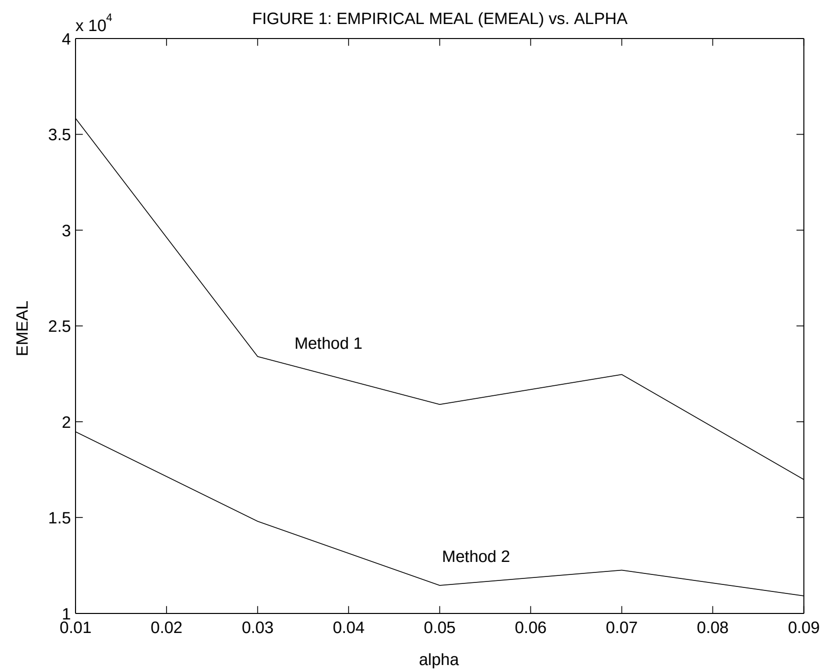


Fig. 2: EMEAL versus K1

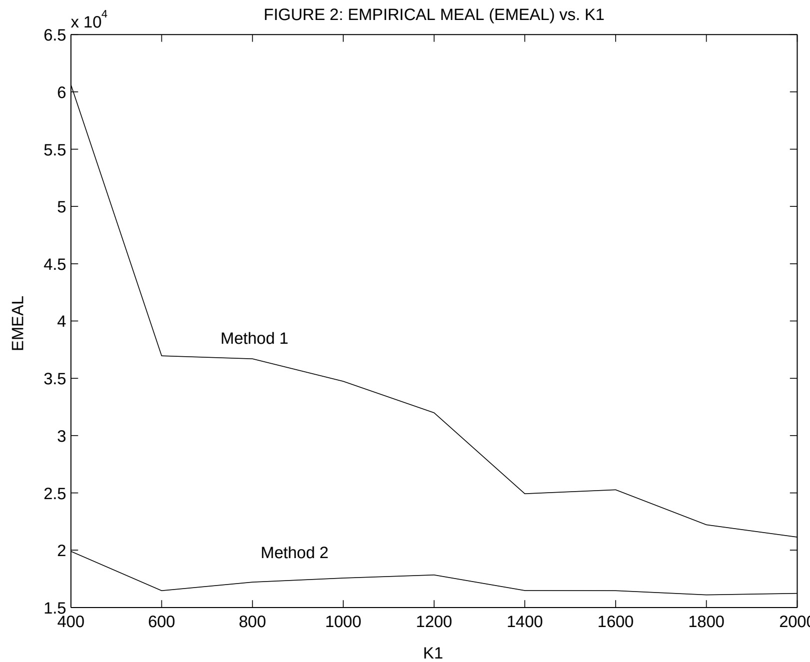


Fig. 3: EMEAL versus P

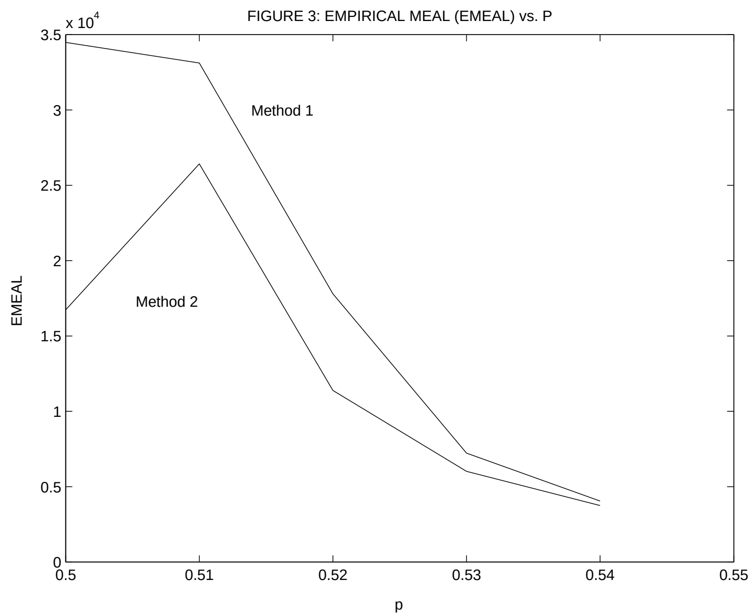


Fig. 4: Number of Loops versus $K_1 = N_1 d^2 / \sqrt{\ln \alpha}$

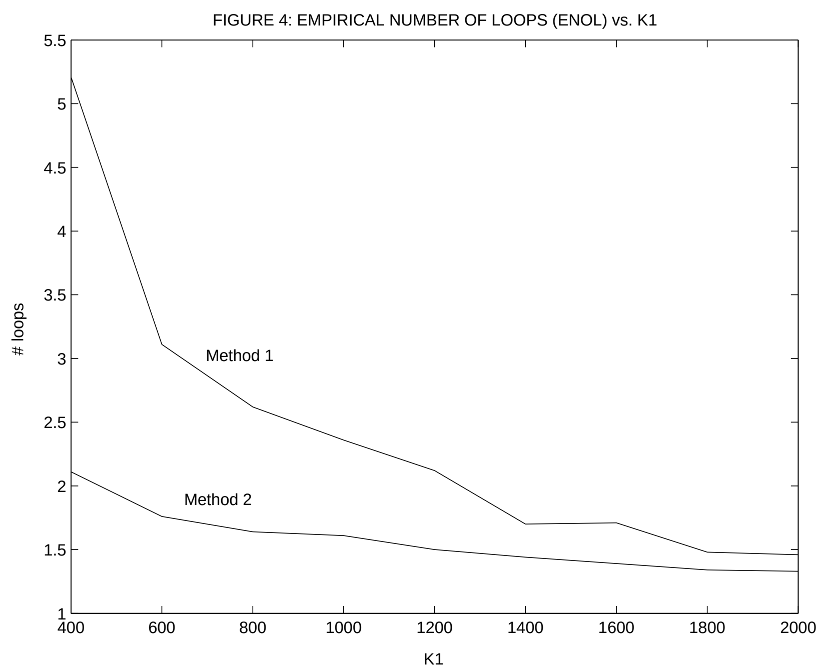


Fig. 5: Error rate versus α

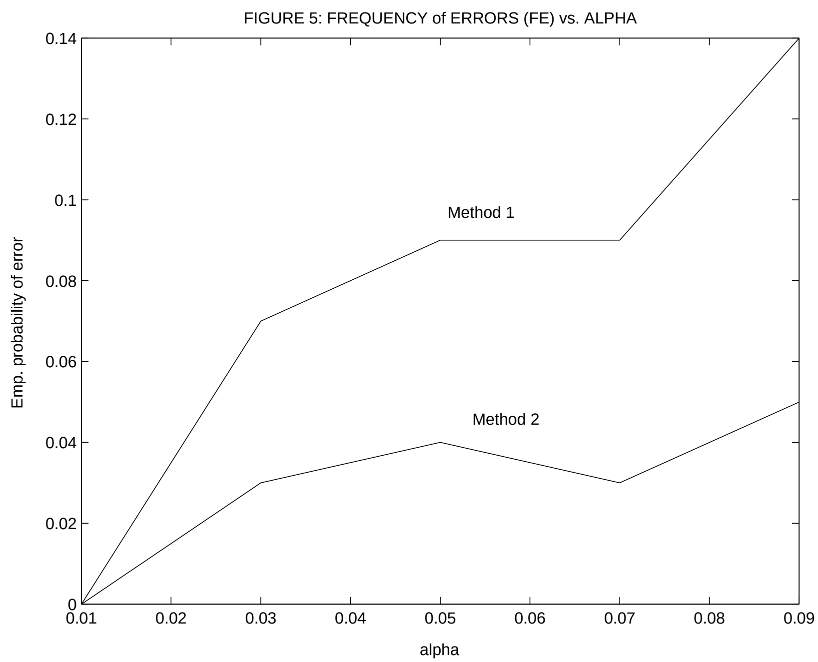
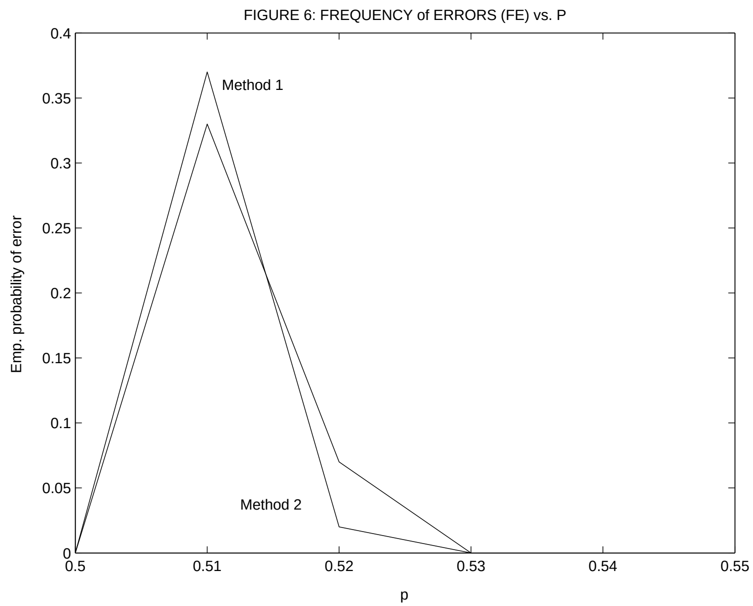


Fig. 6: Error rate versus P



Conclusions

1. Conventional asymptotic expansions of the MEAL in powers of log maximal probability can be inappropriate for discrimination between close hypotheses.

2. Our simulation shows that strategy s^3 is better than strategy s^2.

3. Further work to find a valid expansion for the MEAL of discriminating between close hypotheses seems worthy: use of Le Cam's contiguity and the margin maximization.

4. Suboptimal strategy s^3 is preferable to static discrimination strategies for discrimination between close hypotheses.

5. Use of the MC transition periods for preliminary estimation of true parameters may be fruitful in discrimination between almost deterministic ergodic MC.

REFERENCES

Blot, W.J. and Meeter, D.A. (1973). JASA, **68**, 343, 586-593.

Chernoff, H. (1972). *Sequen. Analys. and Opt. Design*, SIAM.

Cox, D.R. and Hinkley, D.V. (1974). *Theoretical Statistics*, Chapman and Hall, London.

Lai, T.L. (1998). *Information Bounds and Quick Detection of Parameter Changes...*, IEEE IT, **44**, p. 2917–2929.

Malyutov, M.B. and Tsitovich, I.I. (2001). *Second Order Optimal Sequential Discrimination between MC. MMS*, **10**.

Sheffe, H. (1958). *Analysis of Variance*, Wiley, N.Y.

Wald, A. (1947). *Sequential Analysis*, Wiley, New-York.