Recent progress on nonlinear PDEs
due to new results on superdiffusions

1. Trace theory

Goal: to describe all positive solutions of

\[ Lu = \psi(u) \text{ in } E \subset \mathbb{R}^d \]

\( L \) - second order elliptic operator (e.g., \( \Delta \))
\( \psi \in C^2 \) - convex, \( \psi(0) = 0, \psi(u) > 0 \) for \( u > 0 \) + some conditions on behavior at 0 and \( \infty \) (satisfied for \( \psi(u) = u^2, \ 0 \leq 1 \)).

\[ \text{Trace: } \quad \text{Tr}(u) = (\Gamma, \nu) \]

\( \Gamma \) - Borel subset of \( \partial E \)
\( \nu \) - \( \sigma \)-finite measure on \( \partial E \setminus \Gamma \)

[DK 98]: all values of \( \text{Tr}(u) \), 1-1 correspondence between them and \( \delta \)-moderate solutions
\[ u \text{ moderate: } u \leq h \text{ with } Lh = 0 \]
\[ \sigma \text{-moderate: } \exists \text{ moderate } u, \nabla u. \]

Are all solutions \( \sigma \)-moderate?

Mseleti 2002: Yes, for \( \Delta u = u^2 \)
in a bounded smooth domain \( E \)
Tool - Brownian snake of Le Gall.

Dyn. August 2003: Yes, for \( \Delta u = u^\alpha, \ 1 < \alpha \leq 2 \)
Tool - super-Brownian motion.

2. Labeling solutions by measures

Positive labeling, harmonic functions: \( \int k(x, y) v(dy) = h_v(x) \)

\( E \) Poisson finite measures

Labeling moderate solutions: \( u = u_v \) if \( h_v \) is the minimal harmonic majorant of \( u \). Class \( N \).

Labeling of \( \sigma \)-moderate \( u \): \( u = u_v \) if, for some \( v \in N \)
\( v_n \uparrow v \) and \( u v_n \uparrow u \).
3. Lattice structure in the solution space $\mathcal{U}$

Partial order $u \leq v$,

Existence of $\text{Sup } \tilde{U}$ for every $\tilde{U} \subseteq \mathcal{U}$.

$u \oplus v = \text{Sup } \{ w : w \leq u + v \}$.

$\oplus$ preserves sets of moderate and $\delta$-moderate solutions.

If $\tilde{U}$ is closed under $\oplus$, then $\exists u_n \subseteq \tilde{U}$ st.

$u_n \uparrow \text{Sup } \tilde{U}$.

To every Borel $\Gamma \subseteq \partial E$ there corresponds a $\delta$-moderate solution

$u_{\Gamma} = \text{Sup } \{ \text{moderate } u_v : v (\partial E \setminus \Gamma) = 0 \}$

Solution $w_{\Gamma} = \text{Sup } \{ w_K : \text{closed } K \subseteq \Gamma \}$

where $w_K = \text{Sup } \{ u \in \mathcal{U} : u = 0 \text{ on } \partial E \setminus \Gamma \}$

Theorem. If $Tr(u) = (\Gamma, v)$, then

$u_{\Gamma} \oplus u_v \leq u \leq w_{\Gamma} \oplus u_v$. 
Suppose \( u_r = w_r \). Then \( u = u_r \oplus u \) is \( \delta \)-moderate.

**How to prove that \( u_r = w_r \)?**

*Nontrivial part* \( \overline{w}_r \leq \text{const.} \ u_r. \)

Mseleli's idea: (a) upper bound for \( \overline{w}_r \),
(b) lower bound for \( u_r \)
- in terms of a capacity of \( r \).

Part (a) - purely analytical

Part (b) - based on Brownian snake (Mseleli)
or on superdiffusions (Dynkin-Kuznetsov).
\( \mathcal{F}_{\text{int}} = \mathcal{G}(X_D | D' \subset D), \) ("interior")

\( \mathcal{F}_{\text{ext}} = \mathcal{G}(X_D | D'' \supset D), \) ("exterior")

Markov property

\( \mathcal{F}_{\text{int}} \perp \mathcal{F}_{\text{ext}} \)

Absolute continuity property

\( P_x < P_y \) on \( \mathcal{F}_{\text{ext}} \) for \( x, y \in D \)
4. Superdiffusion

Random evolution of a cloud of particles. A state of the cloud - a measure in $E$. Passage to the limit from a branching particle system.

A family of exit measures

$$\left( X^D_q, P^q \right)$$

$$D \subset E$$ initial state

For every smooth $f$ and every $\mu$,

$$- \log P^q \left[ e^{-\langle f, X^D_q \rangle} \right] = \langle u, \mu \rangle$$

where

$$\begin{cases} L u = f(u) & \text{on } D, \\ u = f & \text{on } \partial D. \end{cases}$$

A new tool - measures $N_x$ such that

$$N_x \left( 1 - e^{-Z} \right) = - \log P^q Z$$

for $Z = \sum \langle f_i, X_{D_i} \rangle$

[Inspired by Le Gall's Brownian snake.]

Here $P_x = P_{\delta_x}$, a unit mass at $x$.
5. Range, stochastic boundary value, energy, capacity

Range $\mathcal{D}_E$ - minimal closed set which supports, a.s., $X_D$ for every $D \subseteq \mathcal{E}$ (the area hit by the cloud)

Stochastic boundary value

$SBV_E(u) = \lim \langle u, X_{D_n} \rangle$ where $D_n \subseteq D_{\text{int}}$ and $\cup D_n = E$ (exists for every $u \in \mathcal{U}$ and for every positive harmonic function $h$)

Energy function for $\nu$

$E_x^\nu(v) = \mathbb{E} \int_0^{\tau_E} \psi(E_x(v(t))) dt$

($X, \mathbb{P}_x$ - diffusion with generator $L$,
$\tau_E$ - first exit time from $E$)

$Cap_x(B) = \sup \{ E_x(v): v(B) = 1, v(D_E \setminus B) = 0 \}$
6. Basic inequality

\[ N_x \{ R_E \subset D^*, Z_v \neq 0 \} \geq \text{const.} N_x \{ R_E \subset D^*, Z_v \} \frac{\alpha - 1}{\alpha - 1} \]

\[ \mathcal{E}_x (u) \frac{1}{\alpha - 1} \]

\[ D \in E \text{ smooth, bounded} \]

\[ D^* = \{ x \in \overline{D} : d(x, E^c) > 0 \} = D \cup L \]

\[ V \text{-finite measure on } \partial E \]

\[ Z_v = SB V_E (h_v) \]
The basic inequality follows from the Schwarz inequality in the case $\lambda = 2$ (because $N_x(\mathbb{Z}_V^2) = E_x(v)$).

For $\lambda < 2$, $N_x(\mathbb{Z}_V^d) = \infty$. Hölder's inequality cannot be used. The basic inequality is proved by using relations between superdiffusion and diffusion conditioned by a given exit point from $E$. 