Recent progress on nonlinear PDEs due to new results on superdiffusions

1. Trace theory

Goal: to describe all positive solutions of

$$\boxed{Lu = \psi(u) \text{ in } E \subset \mathbb{R}^d}$$

L - second order elliptic operator (e.g., Δ)

$\psi \in C^1$ - convex, $\psi(0)=0$, $\psi(u)>0$ for $u>0$ + some conditions on behavior at 0 and ∞ (satisfied for $\psi(u)=u^\alpha$, $\alpha>1$).

Trace: $\boxed{Tr(u) = (\Gamma, \nu)}$

Γ - Borel subset of ∂E

ν - σ-finite measure on $\partial E \setminus \Gamma$

[DK 98]: all values of $Tr(u)$, 1-1 correspondence between them and σ-moderate solutions

u moderate: $u \le h$ with $Lh = 0$

u σ-moderate: $\exists$ moderate $u_n \uparrow u$.

Are all solutions σ-moderate?

Mselati 2002: Yes, for $\Delta u = u^2$
in a bounded smooth domain E
Tool - Brownian snake of Le Gall.

Dyn. August 2003: Yes, for $\Delta u = u^\alpha$, $1 < \alpha \le 2$
Tool - super-Brownian motion.

2. Labeling solutions by measures

Labeling positive harmonic functions: $\int_{\partial E} k(x,y)\, \nu(dy) = h_\nu(x)$

Poisson (k), finite measures (ν)

Labeling moderate solutions: $u = u_\nu$ if h_ν is the minimal harmonic majorant of u. Class $\mathcal{N}_1$.

Labeling of σ-moderate u: $u = u_\nu$ if, for some $\nu_n \in \mathcal{N}_1$, $\nu_n \uparrow \nu$ and $u_{\nu_n} \uparrow u$.

3. Lattice structure in the solution space $\mathcal{U}$

Partial order $u \le v$,

Existence of $\operatorname{Sup} \tilde{\mathcal{U}}$ for every $\tilde{\mathcal{U}} \subset \mathcal{U}$.

$u \oplus v = \operatorname{Sup}\{w \le u + v\}$.

$\oplus$ preserves sets of moderate and σ-moderate solutions

If $\tilde{\mathcal{U}}$ is closed under $\oplus$, then $\exists\, u_n \in \tilde{\mathcal{U}}$ s.t.

$$u_n \uparrow \operatorname{Sup} \tilde{\mathcal{U}}.$$

To every Borel $\Gamma \subset \partial E$ there corresponds

σ-moderate solution

$$u_\Gamma = \operatorname{Sup}\{\text{moderate } u_\nu : \nu(\partial E \setminus \Gamma) = 0\}$$

Solution

$$w_\Gamma = \operatorname{Sup}\{w_K : \text{closed } K \subset \Gamma\}$$

where $\quad w_K = \operatorname{Sup}\{u \in \mathcal{U} : u = 0 \text{ on } \partial E \setminus \Gamma\}$

Theorem. If $\operatorname{Tr}(u) = (\Gamma, \nu)$, then

$$u_\Gamma \oplus u_\nu \le u \le w_\Gamma \oplus u_\nu .$$

Suppose $u_\Gamma = w_\Gamma$. Then $u = u_\Gamma \oplus u_\nu$ is δ-moderate.

How to prove that $u_\Gamma = w_\Gamma$?

Nontrivial part $w_\Gamma \le \text{const.}\, u_\Gamma$.

Mselati's idea: (a) upper bound for w_Γ,
(b) lower bound for u_Γ

– in terms of a capacity of Γ.

Part (a) – purely analytical

Part (b) – based on Brownian snake (Mselati)

or on superdiffusions (Dynkin – Kuznetsov).

σ-algebras

$$\mathcal{F}_{\subset D} = \sigma(X_{D'} \,;\, D' \subset D), \quad (\text{"interior"})$$

$$\mathcal{F}_{\supset D} = \sigma(X_{D''} \,:\, D'' \supset D). \quad (\text{"exterior"})$$

Markov property

$$\mathcal{F}_{\subset D} \overset{X_D}{\perp} \mathcal{F}_{\supset D}$$

Absolute continuity property

$$P_x \prec P_y \quad \text{on } \mathcal{F}_{\supset D} \quad \text{for } x, y \in D$$

4. Superdiffusion

Random evolution of a cloud of particles.

A state of the cloud — a measure in E.

Passage to the limit from a branching particle system.

A family of exit measures

$$(X_D, P_\mu)$$

$D \subset E$ initial state

For every smooth f and every μ,

$$-\log P_\mu e^{-\langle f, X_D \rangle} = \langle u, \mu \rangle$$

where
$$\begin{cases} Lu = \psi(u) & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases}$$

A new tool — measures $\mathbb{N}_x$ such that

$$\mathbb{N}_x (1 - e^{-Z}) = -\log P_x Z$$

$$\text{for } Z = \sum \langle f_i, X_{D_i} \rangle$$

[Inspired by Le Gall's Brownian snake.]

Here $P_x = P_{\delta_x}$.

unit mass at x

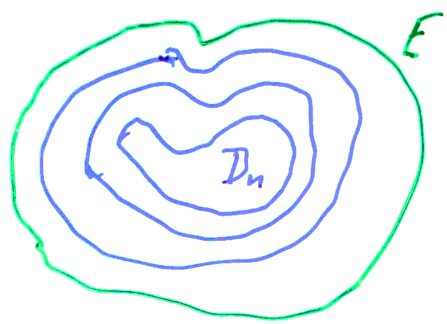
5. Range, stochastic boundary value, energy, capacity

Range $\mathcal{R}_E$ - minimal closed set which supports, a.s., X_D for every $D \subset E$ (the area hit by the cloud)

stochastic boundary value

$SBV_E(u) = \lim \langle u, X_{D_n} \rangle$ where $\bar{D}_n \subset D_{n+1}$ and $\bigcup D_n = E$

(exists for every $u \in \mathcal{U}$ and for every positive harmonic function h)



Energy function for ν

$$\mathcal{E}_x(\nu) = \Pi_x \int_0^{\tau_E} \psi[h_\nu(\xi_t)]\,dt$$

((ξ_t, Π_x) - diffusion with generator L,
τ_E - first exit time from E)

$$\mathrm{Cap}_x(B) = \sup\{\mathcal{E}_x(\nu)^{-1} : \nu(B) = 1, \nu(\partial E \setminus B) = 0\}$$

6. Basic inequality

$$\boxed{\begin{aligned}&\mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu \neq 0\}\\ &\geq \text{const.}\, \mathbb{N}_x\{\mathcal{R}_E \subset D^*, Z_\nu\}^{\alpha/\alpha-1}\, \mathcal{E}_x(\nu)^{-\frac{1}{\alpha-1}}\end{aligned}}$$

$D \subset E$ smooth, bounded

$D^* = \{x \in \bar{D} : d(x, E \setminus D) > 0\}$
$= D \cup L$

ν - finite measure on $\partial D \cap \partial E$

$Z_\nu = SBV_E(h_\nu)$

The basic inequality follows from the Schwarz inequality in the case $\alpha=2$ (because $\mathbb{N}_x(Z_\nu^2)=\mathcal{E}_x(\nu)$

For $\alpha<2$, $\mathbb{N}_x(Z_\nu^\alpha)=\infty$. Hölder's inequality cannot be used. The basic inequality is proved by using relations between superdiffusion and diffusion conditioned by a given exit point from E.