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The Refinability of Step Functions

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Abstract: We will develop necessary and sufficient conditions for the refinability of step functions taking values from **R** to **C**.

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1 Introduction

First the definition of refinability:

Definition 1.1 *A function $f : \boldsymbol{R} \to \boldsymbol{C}$ is 2-refinable if*

$$f(x) = \sum_{l=0}^{N} c_l f(2x - l) \tag{1}$$

where $c_l \in \boldsymbol{C} \ \ \forall \ l \in \{0, \ldots, N\}$

Strictly speaking we should allow the sum to extend over any finite set of integers, but by translating f we may obtain the form (1) with $c_0 \neq 0$ and $c_N \neq 0$.

In particular we will be interested in the refinability of step functions $f : \mathbf{R} \to \mathbf{C}$,

$$f(x) = \sum_{l=1}^{N} k_l \chi_{[l-1,l)} \tag{2}$$

where $N \in \{1, 2, 3, \ldots\}$ and k_l is a complex number $\forall\, l \in \{1, \ldots, N\}$

Without loss of generality we can assume $k_1 \neq 0$ and $k_N \neq 0$. The proof of Theorem 2.1 shows that the value of N must be the same in (1) and (2). For clarity of exposition we assume this from the start. For simplicity, we will assume $k_1 = 1$, since any multiple of a solution to (1) is also a solution.

A simple example of a refinable step function is $f(x) = \chi_{[0,1)} - \chi_{[1,2)}$ where $f(x) = f(2x) + 2f(2x-1) + f(2x-2)$

We will be looking at these functions from an algebraic standpoint as well as their Fourier transforms. In particular, we will examine the relationships between three sets of data that determine these functions: the values that the function takes (the k_l's), the function's refinability constants (the c_l's), and the function's Fourier transform. Using these relations we will come up with a complete classification of these functions.

It should be noted that the result of section 2 is contained in a more general result given in [4]. However, the result given here is obtained in a simpler manner. Furthermore, the results of section 3 are new as are the tables given in section 4.

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2 Classifying Piecewise Constant Functions that are Refinable

The most straightforward way to start classifying refinable step functions is to look at the set of $2N$ algebraic equations that must necessarily be satisfied. Assuming that f satisfies equations (1) and (2), this set of equations is

for equation n, $1 \leq n \leq N$,

$$k_{\lceil n/2 \rceil} = \sum_{l=1}^{n} c_{l-1} k_{n-l+1} \tag{3}$$

and for $N < n \leq 2N$

$$k_{\lceil n/2 \rceil} = \sum_{l=1}^{2N-n+1} c_{N-l+1} k_{n-N+l-1} \tag{4}$$

Using a normalization $k_1 = 1$ one has $2N$ equations and $2N$ variables $(k_2, ..., k_N$ and $c_0, ..., c_N)$. Using equations 1 and $2N$, one can see that $c_0 = 1$ and $c_N = 1$. Furthermore, using just the first N equations, we can see that the c's determine the k's when assuming $k_1 = 1$.

$$k_n = k_{\lceil n/2 \rceil} - (c_1 k_{n-1} + \ldots + c_{n-1} k_1) \quad 2 \leq n \leq N$$

Also, the k's determine the c's, $c_0 = c_N = 1$

$$c_n = k_{\lceil (n+1)/2 \rceil} - (k_{n+1} + c_1 k_n + \ldots + c_{n-1} k_2) \quad 1 \leq n \leq N+1$$

We will now define the Fourier Transform, $\mathcal{F}$, dilation, D^j, and translation, T^j, of a function f as follows

$\mathcal{F}(f(x)) = \hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ixt} dt$

$D^j(f(x)) = \frac{f(2^j x)}{2^{j/2}}$

$T^j(f(x)) = f(x - j)$

Theorem 2.1 *Let $p \in \boldsymbol{C}[z]$, where $\boldsymbol{C}[z]$ is defined as the set of all complex polynomials. Also, we restrict $z = e^{-ix}$, $x \in \boldsymbol{R}$. Then $\frac{p(z)}{ix\sqrt{2\pi}}$ is the Fourier Transform of a refinable step function if and only if $\exists\ q \in \boldsymbol{C}[z]$ such that $p(z^2)/p(z) = q(z)$ and 1 is a zero of $p(z)$.*

Proof. Assume that f satisfies equations (1) and (2). Equation (1) implies

$$\begin{aligned} f(x/2) &= \sum_{l=0}^{N} c_l f(x-l) \\ \frac{1}{\sqrt{2}} D^{-1}(f(x)) &= \sum_{l=0}^{N} c_l T^l(f(x)) \end{aligned}$$

Note that $\hat{D} = D^{-1}$ and $\hat{T}^l = M_{e^{-ilx}}$ (mulitply by e^{-ilx}) Therefore taking the Fourier transform of both sides gives for the left hand side

$$\mathcal{F}(\frac{1}{\sqrt{2}} D^{-1}(f(x))) = \mathcal{F}(f(x/2)) = 2\hat{f}(2x)$$

and for the right hand side

$$\mathcal{F}(\sum_{l=0}^{N} c_l T^l(f(x))) = (\sum_{l=0}^{N} c_l e^{-ilx})\hat{f}(x) = m(x)\hat{f}(x)$$

where $m(x) = \sum_{l=0}^{N} c_l z^l$, $z = e^{-ix}$, and thus $m(x) \in \mathbf{C}[z]$. Equating both sides and dividing by $\hat{f}(x)$ gives

$$\frac{\hat{f}(2x)}{\hat{f}(x)} = \frac{1}{2} m(x) \tag{5}$$

Equation (2) implies

$$\begin{aligned} \hat{f}(x) &= \frac{1}{\sqrt{2\pi}} \sum_{l=1}^{N} k_l \int_{l-1}^{l} e^{-ixt} dt \\ &= \frac{-1}{ix\sqrt{2\pi}} (\sum_{l=1}^{N} k_l (z^l - z^{l-1})) \\ &= \frac{-p(z)}{ix\sqrt{2\pi}} \end{aligned}$$

where

$$p(z) = \sum_{l=1}^{N} k_l (z^l - z^{l-1}) = (z-1) \sum_{l=1}^{N} k_l z^{l-1} \tag{6}$$

Thus $p(z)$ has 1 as a zero and from equation (5) we know

$$\frac{p(z^2)}{p(z)} = m(x) = q(z) \tag{7}$$

The converse is obtained by simply taking the inverse Fourier transform of $\frac{p(z)}{ix\sqrt{2\pi}}$, which will give one a piecewise constant function. Thus the Fourier transform of this function satisfies equation (5) and one gets that it is refinable. QED.

An important thing to note from the proof of this theorem is that the coefficients of $\frac{p(z)}{z-1}$ are the values that function takes, while the coefficients of $q(z)$ are the refinement constants.

Theorem 2.2 *Let $p \in \boldsymbol{C}[z]$. Then $p(z)$ satisfies the conditions of Theorem 2.1 if and only if a.) for every zero, λ, of $p(z)$ with multiplicity m, then λ^2 is also a zero with multiplicity $\geq m$; and b.) 1 is a zero of $p(z)$.*

Proof. Since $p(z)$ is of degree N, write $p(z)$ in terms of its distinct zeroes $\{\lambda_1, \ldots, \lambda_n\}, n \leq N$, each with multiplicity m_l

$$p(z) = k_N \prod_{l=1}^{n} (z - \lambda_l)^{m_l}$$

Furthermore, since p(z) satisfies equation (4) we know that

$$\frac{p(z^2)}{p(z)} = \frac{k_N \prod (z^2 - \lambda_l)^{m_l}}{k_N \prod (z - \lambda_l)^{m_l}} = q(z) \in \mathbf{C}[z]$$

Therefore each multiplicative term in denominator must cancel with a term in the numerator and the theorem is proved. QED.

Furthermore, as a consequence of the lemma, we know that $\lambda_l = e^{2\pi i \frac{p}{q}}$ where p and q are integers.

In summary, a refinable step function is determined by either $K = \{k_1, \ldots, k_N\}$, $C = \{c_0, \ldots, c_N\}$, or the roots of $p(z)$, $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$. Given any one of these sets, we can compute the other two sets.

$K \to p(z)$ which gives Λ by (6) $\to \frac{p(z^2)}{p(z)}$ which gives C by (7)

$C \to$ equations (3) and (4) which gives K

$\Lambda \to k_N \frac{1}{z-1} \prod (z - \lambda_l)^{m_l}$ where $-1 = k_N \prod (-\lambda_l)^{m_l}$ which then gives K by (6)

Theorem 2.2 characterizes the possible sets Λ

3 Miscellaneous Properties of Refinable Step Functions

Theorem 3.1 *For any refinable step function, the refinement coefficients satisfy the following property*

$$c_l = \overline{c_{N-l}} \quad \forall\, l \in \{0 \dots \lfloor N/2 \rfloor\}$$

Proof. Let $\Gamma = \{\gamma_1, \dots, \gamma_N\}$ be the zeroes (not necessarily distinct) of $\frac{p(z^2)}{p(z)}$. Thus we have

$$\frac{p(z^2)}{p(z)} = \prod_{l=1}^{N}(z - \gamma_l) = \sum_{l=0}^{N} \sigma_l(\Gamma) z^l = \sum_{l=0}^{N} c_l z^l$$

where $\sigma_l(\Gamma)$ are the elementary symmetric functions.

$\sigma_0(\Gamma) = \gamma_1\gamma_2 \cdots \gamma_N$

$\sigma_1(\Gamma) = \sum_{l=1}^{N} \gamma_1\gamma_2 \cdots \gamma_{l-1}\gamma_{l+1} \cdots \gamma_N$ where $\gamma_0 = \gamma_{N+1} = 1$

$\vdots$

$\sigma_{N-1}(\Gamma) = \sum_{l=1}^{N} \gamma_l$

$\sigma_N(\Gamma) = 1$

Since $c_l = \sigma_l(\Gamma)$ and $c_0 = 1$ we have $\gamma_1\gamma_2 \cdots \gamma_N = 1$. Furthermore, since $|\gamma_l| = 1$, $\gamma_l^{-1} = \overline{\gamma_l}$. Combining the last two statements gives $\sigma_l(\Gamma) = \overline{\sigma_{N-l}(\Gamma)}$ and thus $c_l = \overline{c_{N-l}}$. QED.

We can also distinguish which polynomials give rise to real valued functions.

Theorem 3.2 *Let f be a refinable step function that's Fourier transform is* $\frac{p(z)}{ix\sqrt{2\pi}}$. *Then f is real valued if and only if for every zero* $e^{i\theta}$ *of p(z) with multiplicity m,* $e^{-i\theta}$ *is a zero as well with the same multiplicity.*

Proof. Assume for $p(z)$ that for every zero $e^{i\theta}$ with multiplicity m, $e^{-i\theta}$ is a zero as well with the same multiplicity. Then for each such pair, one gets

$$(z - e^{i\theta})^m (z - e^{-i\theta})^m = (z^2 - (e^{i\theta} + e^{-i\theta})z + 1)^m = (z^2 - 2\cos\theta z + 1)^m$$

Thus the coefficients of f are real, so f is real valued.

Now assume f is real valued. Thus $\{k_1, \ldots, k_N\}$ are all real numbers. Assume that $e^{i\theta}$ is a zero of $p(z)$, for $e^{i\theta} \neq \pm 1$. Since $(z - e^{i\theta})$ divides $p(z) = (z-1)\sum_{l=1}^{N} k_l z^{l-1}$, this gives $\sum_{l=1}^{N} k_l e^{(l-1)i\theta} = 0$. Since the k_l's are real numbers, when we take the complex conjugate we obtain $p(e^{-i\theta}) = 0$. Similarly, by considering derivatives of p, we can show that the multiplicities are equal. QED.

We can further classify the real valued functions as either even or odd, and can determine when for each.

Theorem 3.3 *Let f be a real valued refinable step function. Then f must be either an even or odd function about N/2. In particular, if*

a.) N=2m

The multiplicity of the zero -1 is odd $\Leftrightarrow$ *f is an even function about N/2*

The multiplicity of the zero -1 is even $\Leftrightarrow$ *f is an odd function about N/2*

b.) N=2m+1

The multiplicity of the zero -1 is odd $\Leftrightarrow$ *f is an odd function about N/2*

The multiplicity of the zero -1 is even $\Leftrightarrow$ *f is an even function about N/2*

Proof. First we see that $k_N = \pm 1$ (assuming $k_1 = 1$) by writing $p(z)$ in terms of its zeros (for this proof each λ_l is not necessarily distinct)

$$\begin{aligned} p(z) = k_N \prod_{l=1}^{N}(z - \lambda_l) &= k_N(z^N + \ldots + (-1)^N \lambda_1 \cdots \lambda_N) \\ &= k_N(z-1)(z^{N-1} + \cdots (-1)^{N-1} \lambda_1 \cdots \lambda_N) \end{aligned}$$

By Theorem 3.2 we know that $\lambda_1 \cdots \lambda_N = \pm 1$. Therefore,

$$k_1 = (-1)^{N-1} k_N \lambda_1 \cdots \lambda_N = \pm k_N = 1$$

In particular, if $N = 2m$, then if the multiplicity of the zero -1 is even, then $k_N = -1$ and if the multiplicity of the zero -1 is odd, then $k_N = 1$. If $N = 2m + 1$, then if the multiplicity of the zero -1 is even, then $k_N = 1$ and if the multiplicity of the zero -1 is odd, then $k_N = -1$. The converses easily follow by using the same logic backwards. We now complete the proof that f must be an even or odd function about $N/2$.

From equations (3) and (4) with $n = 1$ and $n = 2N - 1$ we have

$$\begin{aligned} c_1 k_1 + c_0 k_2 = k_1 \quad &\text{and} \quad c_{N-1} k_N + c_N k_{N-1} = k_N \\ c_1 = \frac{k_1 - k_2}{k_1} \quad &\text{and} \quad c_{N-1} = \frac{k_N - k_{N-1}}{k_N} \end{aligned}$$

By Theorem 3.1 we have $c_1 = c_{N-1}$ and thus

$$\begin{aligned} \frac{k_1 - k_2}{k_1} &= \frac{k_N - k_{N-1}}{k_N} \\ k_1 k_{N-1} &= k_2 k_N \\ \frac{k_1}{k_N} &= \frac{k_2}{k_{N-1}} \end{aligned}$$

Thus $\frac{k_2}{k_{N-1}} = \pm 1$. Using both ratios and equations (3) and (4) with $n = 2$ and $n = 2N - 2$ we get $\frac{k_3}{k_{N-2}} = \frac{k_1}{k_N} = \pm 1$. Continue this process until one gets that all the ratios are equal, and thus f must either be even or odd. QED.

4 Sets of Refinable Step Functions

These tables were computed using the system of algebraic equations detailed in section 2 and solving them in MATLAB.

Table 1: Corresponding Sets of K, C, and Λ of Refinable Step Functions for N=3

k_1, k_2, k_3	c_0, c_1, c_2, c_3	values of t for which $p(e^{2\pi i t}) = 0$
1, 0, -1	1, 1, 1, 1	0, 0, $\frac{1}{2}$
1, 1, 1	1, 0, 0, 1	0, $\frac{1}{3}$, $\frac{2}{3}$
1, -2, 1	1, 3, 3, 1	0, 0, 0
1, 1+i, i	1, -i, i, 1	0, $\frac{1}{4}$, $\frac{1}{2}$
1, 1-i, -i	1, i, -i, 1	0, $\frac{1}{2}$, $\frac{3}{4}$

Table 2: Corresponding Sets of K, C, and Λ of Refinable Step Functions for N=4

k_1, k_2, k_3, k_4	c_0, c_1, c_2, c_3, c_4	values of t for which $p(e^{2\pi it}) = 0$
1, 1, -1, -1	1, 0, 2, 0, 1	0, 0, $\frac{1}{2}$, $\frac{1}{2}$
1, 0, 0, -1	1, 1, 0, 1, 1	0, 0, $\frac{1}{3}$, $\frac{2}{3}$
1, -3, 3, -1	1, 4, 6, 4, 1	0, 0, 0, 0
1, -1, -1, 1	1, 2, 2, 2, 1	0, 0, 0, $\frac{1}{2}$
1, 2, 2, 1	1, -1, 2, -1, 1	0, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$
1, 1, 1, 1	1, 0, 0, 0, 1	0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$
1,i, -1, -i	1, 1-i, 0, 1+i, 1	0, 0, $\frac{1}{4}$, $\frac{1}{2}$
1, -i, -1, i	1, 1+i, 0, 1-i, 1	0, 0, $\frac{1}{2}$, $\frac{3}{4}$
1, $\frac{1}{2}+i\frac{\sqrt{7}}{2}$, $-\frac{1}{2}+i\frac{\sqrt{7}}{2}$, -1	1, $\frac{1}{2}-i\frac{\sqrt{7}}{2}$, -1, $\frac{1}{2}+i\frac{\sqrt{7}}{2}$, 1	0, $\frac{1}{7}$, $\frac{2}{7}$, $\frac{4}{7}$
1, $\frac{1}{2}-i\frac{\sqrt{7}}{2}$, $-\frac{1}{2}-i\frac{\sqrt{7}}{2}$, -1	1, $\frac{1}{2}+i\frac{\sqrt{7}}{2}$, -1, $\frac{1}{2}-i\frac{\sqrt{7}}{2}$, 1	0, $\frac{3}{7}$, $\frac{5}{7}$, $\frac{6}{7}$
1, $\frac{1}{2}-i\frac{\sqrt{3}}{2}$, $\frac{1}{2}-i\frac{\sqrt{3}}{2}$, $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$	1, $\frac{1}{2}+i\frac{\sqrt{3}}{2}$, -1, $\frac{1}{2}-i\frac{\sqrt{3}}{2}$, 1	0, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{5}{6}$
1, $\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $\frac{1}{2}+i\frac{\sqrt{3}}{2}$, $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$	1, $\frac{1}{2}-i\frac{\sqrt{3}}{2}$, -1, $\frac{1}{2}+i\frac{\sqrt{3}}{2}$, 1	0, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{2}{3}$
1, $\frac{2+\sqrt{2}}{2}+i\frac{2-\sqrt{2}}{2}$, $\sqrt{2}+i$, $\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$	1, $-\frac{\sqrt{2}}{2}-i\frac{2-\sqrt{2}}{2}$, $\sqrt{2}$, $-\frac{\sqrt{2}}{2}+i\frac{2-\sqrt{2}}{2}$, 1	0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{5}{8}$
1, $\frac{2-\sqrt{2}}{2}+i\frac{2+\sqrt{2}}{2}$, $-\sqrt{2}+i$, $-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$	1, $\frac{\sqrt{2}}{2}-i\frac{2+\sqrt{2}}{2}$, $-\sqrt{2}$, $\frac{\sqrt{2}}{2}+i\frac{2+\sqrt{2}}{2}$, 1	0, $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$
1, $\frac{2+\sqrt{2}}{2}-i\frac{2-\sqrt{2}}{2}$, $\sqrt{2}-i$, $\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$	1, $-\frac{\sqrt{2}}{2}+i\frac{2-\sqrt{2}}{2}$, $\sqrt{2}$, $-\frac{\sqrt{2}}{2}-i\frac{2-\sqrt{2}}{2}$, 1	0, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{3}{4}$
1, $\frac{2-\sqrt{2}}{2}-i\frac{2+\sqrt{2}}{2}$, $-\sqrt{2}-i$, $-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$	1, $\frac{\sqrt{2}}{2}+i\frac{2+\sqrt{2}}{2}$, $-\sqrt{2}$, $\frac{\sqrt{2}}{2}-i\frac{2+\sqrt{2}}{2}$, 1	0, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$

Table 3: Corresponding Sets of K, C, and Λ of Refinable Step Functions for N=5 (Real Valued functions only)

k_1, k_2, k_3, k_4, k_5	$c_0, c_1, c_2, c_3, c_4, c_5$	values of t for which $p(e^{2\pi it}) = 0$
1, 0, 1, 0, 1	1, 1, -1, -1, 1, 1	0, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{5}{6}$
1, 0, -2, 0, 1	1, 1, 2, 2, 1, 1	0, 0, 0, $\frac{1}{2}$, $\frac{1}{2}$
1, 1, 1, 1, 1	1, 0, 0, 0, 0, 1	0, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$
1, -4, 6, -4, 1	1, 5, 10, 10, 5, 1	0, 0, 0, 0, 0
1, -1, 0, -1, 1	1, 2, 1, 1, 2, 1	0, 0, 0, $\frac{1}{3}$, $\frac{2}{3}$
1, 2, 3, 2, 1	1, -1, 1, 1, -1, 1	0, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{3}$
1, 0, 0, 0, -1	1, 1, 0, 0, 1, 1	0, 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$
1, 1, 0, -1, -1	1, 0, 1, 1, 0, 1	0, 0, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$
1, -2, 0, 2, -1	1, 3, 4, 4, 3, 1	0, 0, 0, 0, $\frac{1}{2}$

Table 4: Corresponding Sets of K, C, and Λ of Refinable Step Functions for N=6 (Real Valued functions only)

$k_1, k_2, k_3, k_4, k_5, k_6$	$c_0, c_1, c_2, c_3, c_4, c_5, c_6$	values of t for which $p(e^{2\pi it}) = 0$
1, 0. -1, -1, 0, 1	1, 1, 1, 2, 1, 1, 1	0, 0, 0, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$
1, -1, 0, 0, -1, 1	1, 2, 1, 0, 1, 2, 1	0, 0, 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$
1, 3, 5, 5, 3, 1	1, -2, 4, -4, 4, -2, 1	0, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{2}{3}$
1, -3, 2, 2, -3, 1	1, 4, 7, 8, 7, 4, 1	0, 0, 0, 0, 0, $\frac{1}{2}$
1, $1-\sqrt{2}$, $2-\sqrt{2}$, $2-\sqrt{2}$, $1-\sqrt{2}$, 1	1, $\sqrt{2}$, $1-\sqrt{2}$, -2, $1-\sqrt{2}$, $\sqrt{2}$, 1	0, $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, $\frac{7}{8}$
1, $1+\sqrt{2}$, $2+\sqrt{2}$, $2+\sqrt{2}$, $1+\sqrt{2}$, 1	1, $-\sqrt{2}$, $1+\sqrt{2}$, -2, $1+\sqrt{2}$, $-\sqrt{2}$, 1	0, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$, $\frac{5}{8}$, $\frac{3}{4}$
1, 1, 1, 1, 1, 1	1, 0, 0, 0, 0, 0, 1	0, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{6}$
1, 1, -2, -2, 1, 1	1, 0, 3, 0, 3, 0, 1	0, 0, 0, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$
1, 2, 3, 3, 2, 1	1, -1, 1, 0, 1, -1, 1	0, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$
1, 2, 2, 2, 2, 1	1, -1, 2, -2, 2, -1, 1	0, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{1}{2}$, $\frac{3}{5}$, $\frac{4}{5}$
1, 1, 1, -1, -1, -1	1, 0, 0, 2, 0, 0, 1	0, 0, $\frac{1}{3}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{2}{3}$
1, -1, 1, -1, 1, -1	1, 2, 0, -2, 0, 2, 1	0, 0, $\frac{1}{6}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{5}{6}$
1, -2, 1, -1, 2, -1	1, 3, 3, 2, 3, 3, 1	0, 0, 0, 0, $\frac{1}{3}$, $\frac{2}{3}$
1, 2, 1, -1, -2, -1	1, -1, 3, -2, 3, -1, 1	0, 0, $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{2}{3}$
1, 1, 0, 0, -1, -1	1, 0, 1, 0, 1, 0, 1	0, 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{3}{4}$
1, 0, 0, 0, 0, -1	1, 1, 0, 0, 0, 1, 1	0, 0, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$
1, -5, 10, -10, 5, -1	1, 6, 15, 20, 15, 6, 1	0, 0, 0, 0, 0, 0
1, -1, -2, 2, 1, -1	1, 2, 3, 4, 3, 2, 1	0, 0, 0, 0, $\frac{1}{2}$, $\frac{1}{2}$

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