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***The Magnus Representation of the Torelli
Group $I_{g,1}$***

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THE MAGNUS REPRESENTATION OF THE TORELLI GROUP $\mathcal{I}_{g,1}$

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ABSTRACT. The Magnus representations r_k are defined on the Johnson filtration of the mapping class group. We focus on $r_2 : \mathcal{I}_{g,1} \rightarrow \mathrm{GL}_{2g}(\mathbb{Z}[H])$, where $\mathcal{I}_{g,1}$ is the Torelli subgroup of the mapping class group and $H = H_1(S_{g,1})$ is the first homology of the surface. After restricting r_2 to the Johnson kernel $\mathcal{K}_{g,1}$, we classify all relations between pairs of Dehn twists in the image.

1. INTRODUCTION

Let $S_{g,1}$ be a compact oriented surface of genus g with one boundary component. We define the mapping class group of $S_{g,1}$ (relative to the boundary) to be $\mathcal{M}_{g,1} = \mathrm{Homeo}(S_{g,1}) / \sim$, where \sim is the isotopy equivalence relation on (orientation-preserving) homeomorphisms of $S_{g,1}$, and we require that all homeomorphisms and isotopies fix the boundary $\partial S_{g,1}$ pointwise. The structure of the mapping class group has important implications in several areas of mathematics, such as the study of Teichmüller space and low-dimensional topology. Despite various results of Johnson and Morita, this structure is still not well understood.

The Torelli group $\mathcal{I}_{g,1}$ is defined to be the subgroup of the mapping class group $\mathcal{M}_{g,1}$ that acts trivially on the first homology of the surface. In [3] Johnson determined the abelianization of $\mathcal{I}_{g,1}$ and defined a series of homomorphisms corresponding to the Johnson filtration of the mapping class group. The Johnson filtration is a sequence of subgroups $\mathcal{M}(1) \geq \mathcal{M}(2) \geq \dots$ such that $\mathcal{M}(1) = \mathcal{M}_{g,1}$, $\mathcal{M}(2) = \mathcal{I}_{g,1}$, and for each $k \geq 2$, $\mathcal{M}(k+1)$ is the kernel of the Johnson homomorphism τ_k from $\mathcal{M}(k)$ to a certain abelian quotient. Johnson showed in [3] that the “Johnson kernel” $\mathcal{M}(3) = \mathcal{K}_{g,1}$ is precisely the subgroup generated by Dehn twists about separating curves in $S_{g,1}$. Morita studied these Johnson homomorphisms further in [4] and [5], but there are many open questions about this filtration. In particular, very little is known about the structure of the Johnson kernel $\mathcal{M}(3) = \mathcal{K}_{g,1}$, although Biss and Farb [2] have proven that $\mathcal{K}_{g,1}$ is not finitely generated for $g \geq 2$.

There is an important sequence of representations of the terms of the Johnson filtration. For each natural number k , there is the k^{th} Magnus representation $r_k : \mathcal{M}(k) \rightarrow \mathrm{GL}_{2g}(\mathbb{Z}[N_k])$, where N_k is the k^{th} nilpotent quotient of the fundamental group of the surface $S_{g,1}$. These representations were originally defined using Fox calculus, but Suzuki demonstrated in [7] that there is an equivalent topological definition, which we will use throughout this paper. Given a homeomorphism of $S_{g,1}$, we can lift it to a homeomorphism of the k^{th} nilpotent covering space of $S_{g,1}$ and then look at its action on the first homology of this cover, which is isomorphic to $\mathbb{Z}[N_k]^{2g}$. Note that r_1 is simply the symplectic representation of the mapping class group, $r_1 : \mathcal{M}_{g,1} \rightarrow \mathrm{Sp}_{2g}(\mathbb{Z})$. We are most interested in $r_2 : \mathcal{I}_{g,1} \rightarrow \mathrm{GL}_{2g}(\mathbb{Z}[H])$, where $H = N_2$ is the first homology of $S_{g,1}$. Using a modification of Suzuki’s method in [6] of defining a “higher intersection form” on homology elements of the abelian cover, we derive simple formulae describing the restriction of r_2 to $\mathcal{K}_{g,1}$.

Suzuki ([6], Corollary 4.4) characterized when the commutator of two Dehn twists around separating curves $[T_{\gamma_1}, T_{\gamma_2}]$ is in $\ker r_2$. Our methods rederive this result, and they also yield

the following extension, which states that no relations besides commutation can arise between pairs of Dehn twists in the image of the Magnus representation.

Theorem 5.2. *Suppose that γ_1 and γ_2 are separating curves on $S_{g,1}$ with lifts $c_1, c_2 \in \mathbb{Z}[H]^{2g}$ such that $\langle c_1, c_2 \rangle \neq 0$. Then there are no relations between $r_2(T_{\gamma_1})$ and $r_2(T_{\gamma_2})$ in $\mathrm{GL}_{2g}(\mathbb{Z}[H])$, i.e. no nontrivial word in T_{γ_1} and T_{γ_2} is in $\ker r_2$.*

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2. SETUP

We first fix a set of generators for the free group $\Gamma_1 = \pi_1(S_{g,1})$ as shown in Figure 1. We fix a basepoint $b \in \partial S_{g,1}$, and choose $A_1, \dots, A_g, B_1, \dots, B_g$ to be loops based at b . Note that all the A_i, B_i freely generate $\pi_1(S_{g,1})$; that if a_i, b_i are the homology classes of A_i, B_i respectively, then $a_1, \dots, a_g, b_1, \dots, b_g$ form a symplectic basis for $H_1(S_{g,1})$; and that the product of commutators $[A_1, B_1] \cdots [A_g, B_g]$ is a loop around the boundary component.

Let $p : \widehat{S} \rightarrow S_{g,1}$ be the universal abelian cover of $S_{g,1}$; that is, the regular covering space corresponding to the commutator subgroup of Γ_1 . A homeomorphism of $S_{g,1}$ lifts to a homeomorphism of \widehat{S} (though not uniquely), and we want to examine the action of this lifted homomorphism on the homology of \widehat{S} . We can lift our basis of Γ_1 to a $\mathbb{Z}[H]$ -module basis of $H_1(\widehat{S}, p^{-1}(b); \mathbb{Z}) \approx \mathbb{Z}[H]^{2g}$, where $H = H_1(S_{g,1}; \mathbb{Z})$, as follows.

Choose a point $\widehat{b} \in p^{-1}(b) \subset \partial \widehat{S}$ that is a lift of our basepoint $b \in \partial S_{g,1}$. For each $i = 1, \dots, g$, define $\alpha_i \in H_1(\widehat{S}, p^{-1}(b))$ as the unique lift of A_i starting at \widehat{b} , and similarly define β_i to be the lift of B_i starting at \widehat{b} . Each of these arcs must have its endpoints in $p^{-1}(b)$, so each describes an element of $H_1(\widehat{S}, p^{-1}(b))$. We will complete our description of $H_1(\widehat{S}, p^{-1}(b))$ after describing the deck transformations of the abelian cover.

The group of deck transformations of $\widehat{S} \rightarrow S_{g,1}$ is isomorphic to \mathbb{Z}^{2g} , the abelianization of Γ_1 , but we can actually find an identification with H itself. Define the deck transformation a_i

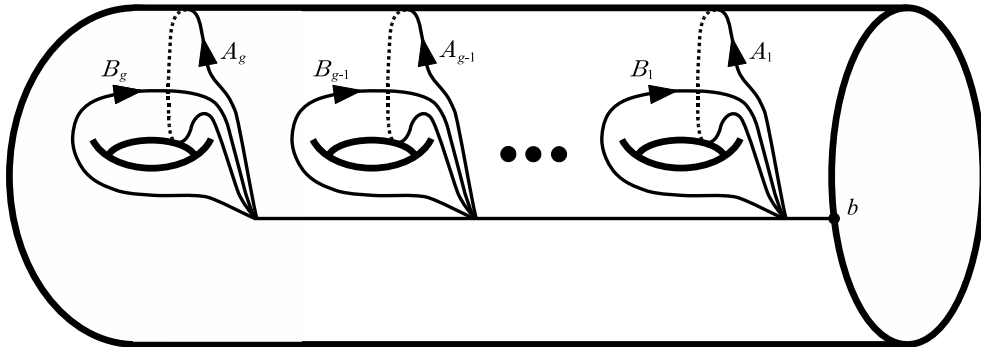


FIGURE 1. A basis of $\pi_1(S_{g,1})$

to be that which translates the tail of the arc α_i (that is, \widehat{b}) to its head, and similarly let b_i be the deck transformation translating \widehat{b} to the head of β_i . Then the obvious identification of this deck transformation a_i with the element $a_i \in H$ gives an isomorphism to H .

This gives an isomorphism between $H_0(p^{-1}(b))$ and $\mathbb{Z}[H]$: $H_0(p^{-1}(b))$ is generated by the points in $p^{-1}(b)$, so for every element $h \in H$, we can identify the point $h(\widehat{b}) \in p^{-1}(b)$ with the element $h \in \mathbb{Z}[H]$ (where $h(\widehat{b})$ is the image of \widehat{b} under the deck transformation h).

Viewing the group of deck transformations as isomorphic to H gives rise to a $\mathbb{Z}[H]$ -action on $H_1(\widehat{S}, p^{-1}(b))$: given $c \in H_1(\widehat{S}, p^{-1}(b))$ and $h \in H$, define $h \cdot c$ to be $h(c)$, the image of γ under the deck transformation h . This H -action extends linearly to a $\mathbb{Z}[H]$ -action. Under this $\mathbb{Z}[H]$ -action, we can regard $H_1(\widehat{S}, p^{-1}(b))$ as a $\mathbb{Z}[H]$ -module of rank $2g$, with basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$.

We can also examine the long exact sequence of homology for the pair $(\widehat{S}, p^{-1}(b))$. The only nonzero section of this sequence is

$$0 \rightarrow H_1(\widehat{S}) \rightarrow H_1(\widehat{S}, p^{-1}(b)) \xrightarrow{\partial} H_0(p^{-1}(b)) \xrightarrow{\varepsilon} H_0(\widehat{S}) \rightarrow 0$$

where ε is the augmentation map $\varepsilon : \mathbb{Z}[H] \rightarrow \mathbb{Z}$ and ∂ is the boundary map on homology.

The Magnus representation $r_2 : \mathcal{I}_{g,1} \rightarrow \mathrm{GL}_{2g}(\mathbb{Z}[H])$ was originally defined using Fox calculus (see e.g. [1]). We use the topological definition given by Suzuki in [7]. An element of $\mathcal{M}(g, 1)$ can be represented by a homeomorphism f of $S_{g,1}$. This lifts uniquely to a homeomorphism \tilde{f} of \widehat{S} , if we require that \tilde{f} fix \widehat{b} . We then examine the action of \tilde{f} on $H_1(\widehat{S}, p^{-1}(b))$. This action is obviously \mathbb{Z} -linear, but unfortunately it is not $\mathbb{Z}[H]$ -linear — it is twisted by the action of f on H . We thus restrict to $\mathcal{I}_{g,1}$, ensuring that the lifts \tilde{f} will act $\mathbb{Z}[H]$ -linearly on $H_1(\widehat{S}, p^{-1}(b))$. Since $H_1(\widehat{S}, p^{-1}(b)) \approx \mathbb{Z}[H]^{2g}$, this action of \tilde{f} can be used to define the Magnus representation $r_2 : \mathcal{I}_{g,1} \rightarrow \mathrm{GL}_{2g}(\mathbb{Z}[H])$.

3. VISUALIZING THE ABELIAN COVER

While it is possible to use this topological definition without ever visualizing the 2-manifold with boundary \widehat{S} , a mental picture can be very useful to ground the algebraic concepts in topological intuition. It also simplifies some calculations, and more importantly, makes the results more illuminating. We give the most useful model that we know for visualizing the abelian cover of $S_{g,1}$. The following description will agree naturally with the description of first homology in the previous section.

In general, just as S_g can be gotten from $S_{g,1}$ by filling in the boundary, the abelian cover of S_g will be just \widehat{S} with all the lifts of the boundary filled in. (If $i : S_{g,1} \rightarrow S_g$ is the obvious inclusion and $i_* : \Gamma_1 \rightarrow \pi_1(S_g)$ the induced map on the fundamental groups, then $\ker i_* \leq [\Gamma_1, \Gamma_1]$.) In the case $g = 1$, \widehat{S} has a simple representation; it is simply the plane \mathbb{R}^2 with a disk removed at every point of the integer lattice \mathbb{Z}^2 . Here, we see that filling in the holes gives the simply-connected \mathbb{R}^2 , but this only happens when $g = 1$, since the abelian cover of $S_{1,0}$ is also its universal cover. For the case $g = 2$, we already have a much more complicated picture for \widehat{S} , the abelian cover of $S_{2,1}$ (Figure 2). Although our description of \widehat{S} works in general for arbitrary g , it is not qualitatively different for higher g , so we will illustrate it by diagrams corresponding to either $g = 1$ or $g = 2$. (Note that the resulting picture as constructed lives in four dimensions, of which we can depict only two at a time.) We now go through the construction of the model shown in Figure 2. The surface is built up by attaching strips to the edges of discs in a specific way.

Start with an integer lattice indexed by \mathbb{Z}^{2g} , and label the $2g$ directions, or factors, by $a_1, \dots, a_g, b_1, \dots, b_g$. Take a regular $4g$ -gon, label its edges in order according to the word

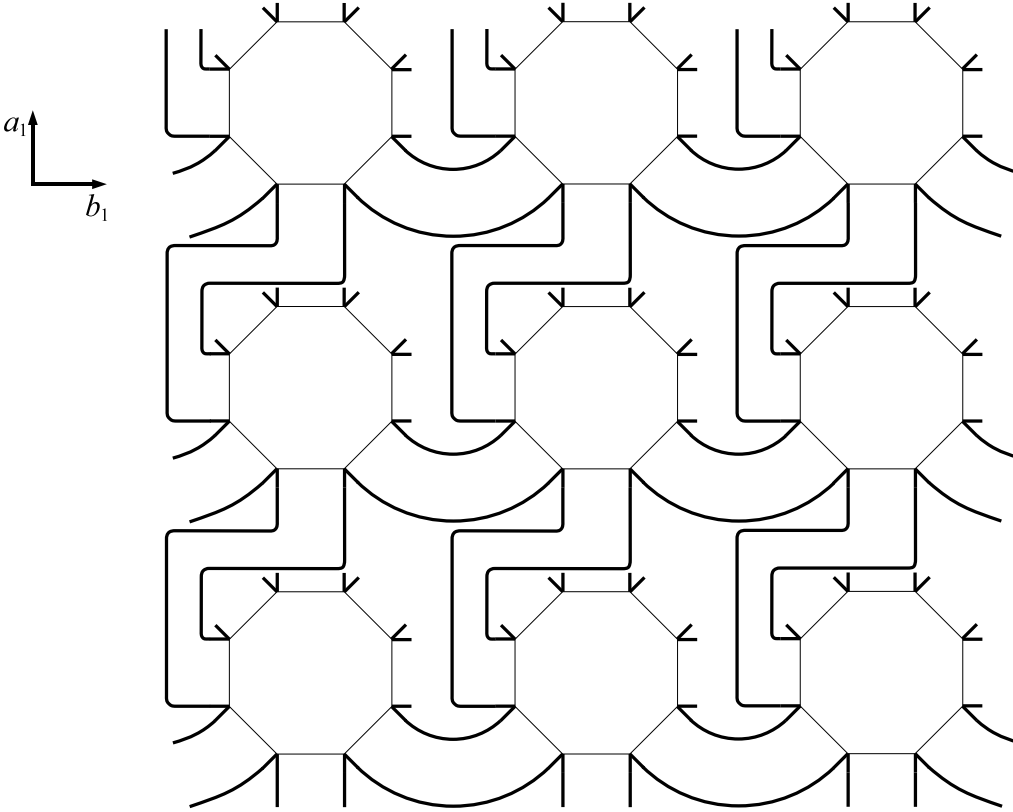


FIGURE 2. A partial depiction of the abelian cover of $S_{2,1}$

$[x_1, y_1^{-1}] \cdots [x_g, y_g^{-1}]$, and mark the vertex between the adjacent edges y_g and x_1 , as in Figure 3. Now place a disjoint copy of this $4g$ -gon at every point of the \mathbb{Z}^{2g} lattice. All of the marked points will be lifts of the basepoint $b \in S_{g,1}$. In the integer lattice there is one polygon at the origin; the marked point in this polygon will be the specific lift \hat{b} .

We now attach the ribbons, or strips, which we think of as long thin rectangles which will be attached to sides of the polygons at their ends. We have labeled each direction in the lattice by a_i or b_i . Attach a ribbon to the x_1 side of each polygon, and attach the other end to the x_1^{-1} side of its neighboring polygon in the a_1 direction. Continue for the other sides; there will be a ribbon from the x_i side of each polygon to the x_i^{-1} side of its neighbor in the a_i direction, and similarly from the y_i side to the y_i^{-1} side of the neighbor in the b_i direction. This gives the picture of Figure 2. None of the ribbons should twist; the polygons all have the same orientation as the original $4g$ -gon, and the ribbons are attached so that all their orientations agree, giving us an oriented surface with boundary.

We have now completed the description of the surface \hat{S} , but we can describe a few more features of it. The group of deck transformations is isomorphic to H ; its generators are given by moving the entire lattice rigidly by one unit in each of the $2g$ directions. We can associate the shift in the a_1 direction with $a_1 \in H$ itself, and so on. The identification of $H_0(p^{-1}(b))$ with $\mathbb{Z}[H]$ is then immediate, since that group is generated by all the marked points making up $p^{-1}(b)$. Given a particular point $p \in p^{-1}(b)$, there is a unique deck transformation k that translates \hat{b} to p , and we identify p with the generator $h \in \mathbb{Z}[H]$. To see that the above description of a surface indeed gives the abelian cover of $S_{g,1}$, consider the quotient surface

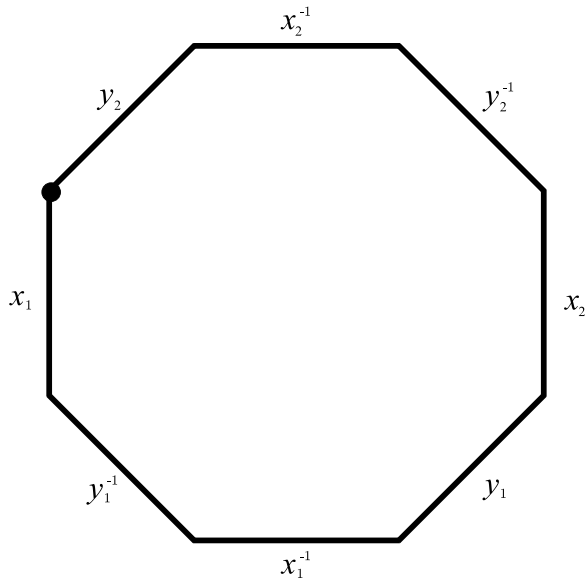


FIGURE 3. The labeling on the $4g$ -gon

given by quotienting out by the action of H described above. This quotient surface is simply a single octagon labeled as above by the word $[x_1, y_1^{-1}] \cdots [x_g, y_g^{-1}]$, with ribbons attached connecting the x_i sides to the x_i^{-1} sides and the y_i sides to the y_i^{-1} sides, and this is a well-known description of $S_{g,1}$.

The main object of our study is $H_1(\widehat{S}, p^{-1}(b))$, and again we can find explicit realizations of its elements. For example, α_1 is represented by an arc running from \widehat{b} to the x_1 side of the polygon that contains it, along the ribbon there in the a_1 direction, and then ending at the basepoint a_1 on the neighboring polygon. The translation $h\alpha_1$ is similar, but it starts at the point h rather than \widehat{b} . Note that $-\alpha_1$ must be the additive inverse of α_1 in relative homology; this means that the arc that begins at \widehat{b} , travels in the negative a_1 direction, and ends at the basepoint a_1^{-1} is *not* $-\alpha_1$, but $-a_1^{-1}\alpha_1$. The other generators of $H_1(\widehat{S}, p^{-1}(b))$ are realized similarly.

A Dehn twist T_γ about a separating curve γ is often called a BSCC map (a separating curve is also called a “bounding simple closed curve”). The lift of a BSCC map to \widehat{S} has a nice realization. The other common type of generator of $\mathcal{I}_{g,1}$ is a bounding pair map, of the form $T_{\gamma_1}T_{\gamma_2}^{-1}$, where γ_1 and γ_2 are homologous and non-zero in homology; it seems that the lift of such a map has no correspondingly simple realization. This is the reason that our results focus on BSCC maps and on $\mathcal{K}_{g,1}$, which is generated by BSCC maps.

The lift of a separating curve γ is a curve c , while the lift of a non-separating curve is an arc. Furthermore, c is disjoint from all its copies hc for $h \in H$. (Any curve, not necessarily simple, which is considered as an element of $\pi_1(S_{g,1}) = \Gamma_1$ lies within $[\Gamma_1, \Gamma_1]$, lifts to a curve. In contrast, the second claim depends on the fact that γ is simple; $[A_1^2, B_1^2] \in \Gamma_1$, which cannot be realized simply, lifts to a curve c' such that c' and a_1b_1c' are never disjoint. See Lemma 4.3.) To visualize \widetilde{T}_γ , we simply take all the disjoint curves hc over $h \in H$; \widetilde{T}_γ will be the simultaneous twist around all of them. We can calculate the effect of this lifted twist on $H_1(\widehat{S}, p^{-1}(b))$ quite easily. If d is the arc or curve in question, simply trace along d ; for each intersection with a lift hc , add or subtract $hc \in H_1(\widehat{S}, p^{-1}(b))$, depending on the orientation of the intersection.

(This is discussed and simplified in Proposition 4.4.) Figure 4 shows the simplest example: \tilde{T}_δ , the lift of the Dehn twist about δ , the boundary of $S_{1,1}$.

To calculate the lift of a bounding pair map $T_\gamma T_\gamma^{-1}$, we would like to realize \tilde{T}_γ as before. This is difficult, though, since T_γ is not in $\mathcal{I}_{g,1}$. Since γ is now nonseparating, its lift $c \in H_1(\widehat{S}, p^{-1}(b))$ will be an arc. If $[\gamma]$ is the homology of γ , then the head of c will coincide with the tail of $[\gamma]c$. Considering all lifts hc of γ , we find that rather than a collection of disjoint curves, we have a collection of disjoint infinite lines. Together, these lines cut \widehat{S} into slices; each slice is characterized by the number of times a path from \widehat{b} to a point in that slice must cross the lifts hc . For example, in Figure 5 we see that all the lifts of A_1 , the vertical lines, cut \widehat{S} into strips characterized by the b_1 component of a location in the lattice. A *Dehn slide* along such a line is defined as the homeomorphism of \widehat{S} given by cutting along the line, then sliding one side along the other until one end of hc has been translated to the other end. It is clear that \tilde{T}_γ is just the simultaneous Dehn slide along all our lines, but from the definition we see that the resulting map is not particularly nice; notably, slices are translated along themselves by a distance corresponding to their distance from the origin. Neither $\partial\widehat{S}$ nor $p^{-1}(b)$ is fixed pointwise, so the resulting map is not $\mathbb{Z}[H]$ -linear. We see that $\tilde{T}_\gamma(hd) = T_\gamma(h)\tilde{T}_\gamma(d)$; the action of \tilde{T}_γ on $H_1(\widehat{S}, p^{-1}(b))$ is twisted by the action of T_γ on H . (This suggests that there is no formula for the effects of bounding pair maps corresponding to Proposition 4.4 for BSCC maps.) It is still possible to compute the effects of a lifted bounding pair map manually, but care must be taken not to rely on the linearity of the two composed maps.

4. HIGHER INTERSECTION FORMS

Following the method of Suzuki in [6], we define a $\mathbb{Z}[H]$ -valued pairing for elements of $H_1(\widehat{S}, p^{-1}(b))$ analogous to the algebraic intersection number of two elements of $H_1(S_{g,1})$. We

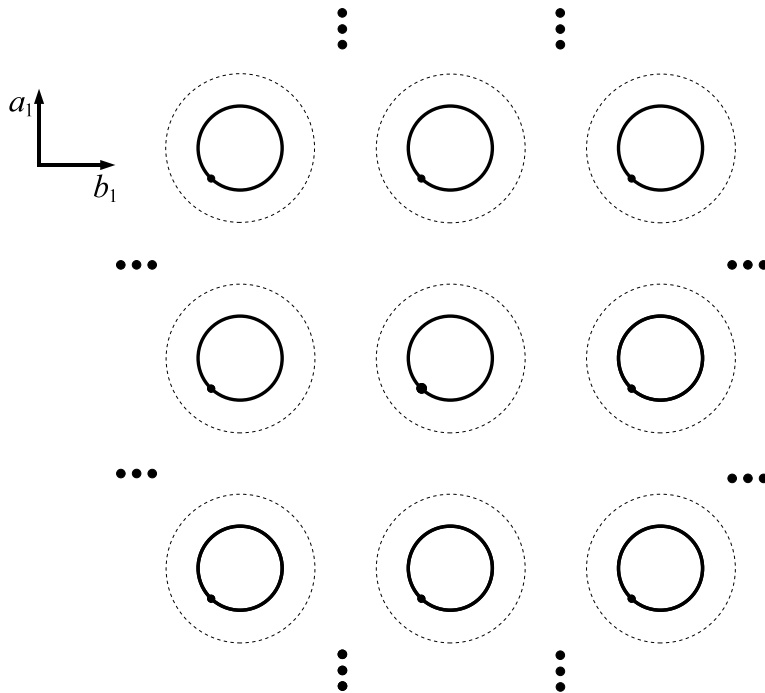


FIGURE 4. The lifts of δ to the abelian cover of $S_{1,1}$

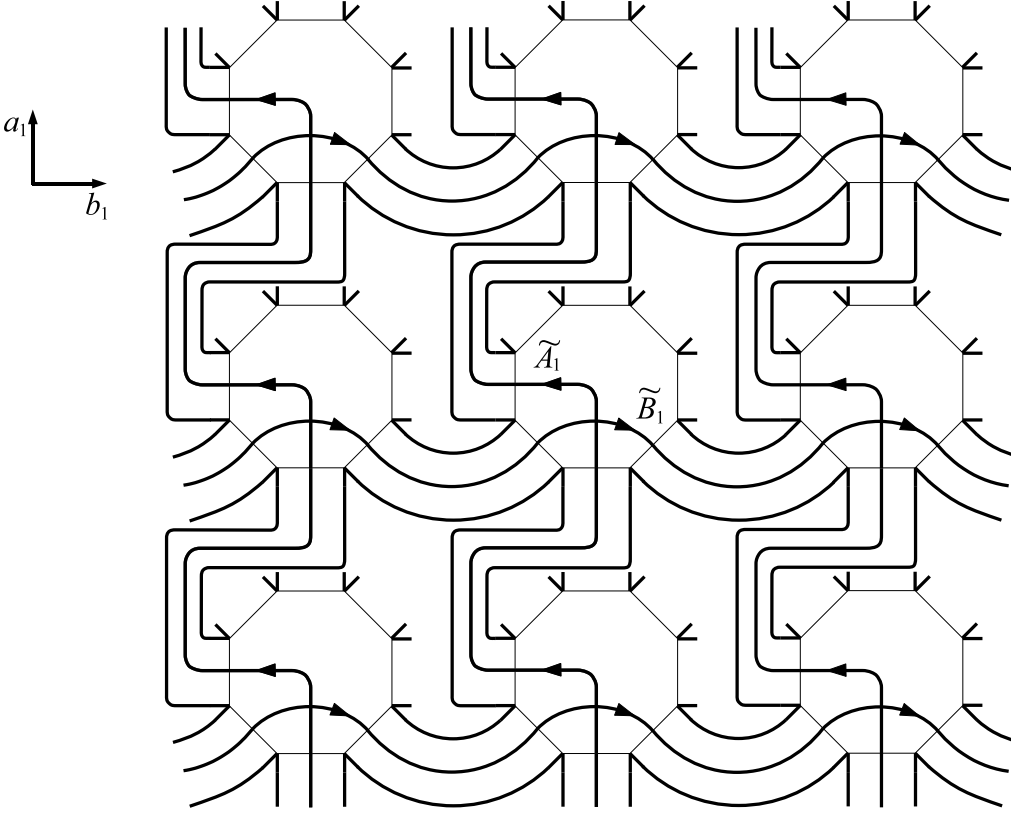


FIGURE 5. The lifts of A_1 and B_1 to the abelian cover of $S_{2,1}$

then list some properties of this form and give applications to describing the kernel and image of r_2 .

We first need to define the (\mathbb{Z} -valued) intersection number of two elements of $H_1(\widehat{S}, p^{-1}(b))$. Any element of $H_1(\widehat{S}, p^{-1}(b))$ can be realized as a linear combination of closed curves in \widehat{S} and arcs in \widehat{S} with endpoints in $p^{-1}(b)$. Any pair of curves or a curve and an arc can be homotoped so that they only intersect transversely, and then the orientation of \widehat{S} gives a natural notion of algebraic intersection number. However, α_1 and β_1 , for example, are two arcs in $H_1(\widehat{S}, p^{-1}(b))$ that share one endpoint, so their intersection number is not well-defined by the above procedure. In order to define their algebraic intersection number, we need to move the basepoint of one arc slightly; there are two different ways of doing so, and we will keep track of the resulting differences in the algebraic intersection number. This will give us two \mathbb{Z} -valued bilinear forms on $H_1(\widehat{S}, p^{-1}(b)) \times H_1(\widehat{S}, p^{-1}(b))$.

We now formalize the above discussion. Let $b' \neq b$ be a second basepoint in $\partial S_{g,1}$, and let γ_1, γ_2 be the two arcs from b to b' contained in $\partial S_{g,1}$, where γ_1 is chosen to have orientation consistent with that of $S_{g,1}$. For $i = 1, 2$ let ξ_i be the lift of γ_i to \widehat{S} based at \widehat{b} . Then we have isomorphisms $\varphi_i : H_1(\widehat{S}, p^{-1}(b)) \rightarrow H_1(\widehat{S}, p^{-1}(b'))$ defined by $\varphi_i(x) = x + \partial x \cdot \xi_i$.

Because there is an induced orientation of \widehat{S} from $S_{g,1}$, we already have a \mathbb{Z} -bilinear algebraic intersection form defined on $H_1(\widehat{S}, p^{-1}(b)) \times H_1(\widehat{S}, p^{-1}(b'))$, and we denote this form by (\cdot, \cdot) . We define two bilinear forms $(\cdot, \cdot)_i$ on $H_1(\widehat{S}, p^{-1}(b)) \times H_1(\widehat{S}, p^{-1}(b))$ by $(c, d)_i = (c, \varphi_i(d))$,

$i = 1, 2$. Now, define two forms $\langle \cdot, \cdot \rangle_i$ for $i = 1, 2$ on $H_1(\widehat{S}, p^{-1}(b)) \times H_1(\widehat{S}, p^{-1}(b))$ by

$$(1) \quad \langle c, d \rangle_i = \sum_{h \in H} (c, hd)_i h \in \mathbb{Z}[H]$$

Both forms $\langle \cdot, \cdot \rangle_i$ are pseudo-bilinear; that is, they are linear in the first term and anti-linear in the second: $\langle hc + c', d \rangle_i = h\langle c, d \rangle_i + \langle c', d \rangle_i$ and $\langle c, hd + d' \rangle_i = h^{-1}\langle c, d \rangle_i + \langle c, d' \rangle_i$. (This is immediate from the definition and the linearity of the intersection forms $(\cdot, \cdot)_i$.) We call these forms higher intersection forms because they lift the normal algebraic intersection form (\cdot, \cdot) on H . In other words, if $p_* : H_1(\widehat{S}, p^{-1}(b)) \rightarrow H$ is the map on first homology induced by the covering map $p : \widehat{S} \rightarrow S_{g,1}$, then $\varepsilon(\langle c, d \rangle_i) = (p_*(c), p_*(d))$ for all $c, d \in H_1(\widehat{S}, p^{-1}(b))$.

The following lemma will be very important to us:

Lemma 4.1. *Both forms $\langle \cdot, \cdot \rangle_i$ are nondegenerate; that is, for $x \in \mathbb{Z}[H]^{2g} \setminus \{0\}$ and $i = 1$ or 2 , there exists $y \in \mathbb{Z}[H]^{2g}$ such that $\langle x, y \rangle_i \neq 0$.*

Proof. This follows immediately from the fact that the forms $(\cdot, \cdot)_i$ are nondegenerate, which is a simple computational exercise. \square

At this point, it is useful to define the isomorphism $\bar{\cdot} : \mathbb{Z}[H] \rightarrow \mathbb{Z}[H]$ given by $\bar{h} = h^{-1}$ and linear extension to all of $\mathbb{Z}[H]$. The following proposition tells us the difference between the two higher intersection forms.

Proposition 4.2. *When either c or d is a curve in \widehat{S} (that is, when c or d is in $\ker \partial$), $\langle c, d \rangle_1 = \langle c, d \rangle_2$. In general, $\langle c, d \rangle_2 - \langle c, d \rangle_1 = \partial c \bar{\partial} d$.*

Proof. Note that

$$\begin{aligned} (x, y)_2 - (x, y)_1 &= (x, \varphi_2(y)) - (x, \varphi_1(y)) \\ &= (x, (y + \partial y \cdot \xi_1) - (y + \partial y \cdot \xi_2)) \\ &= (x, \partial y (\xi_2 - \xi_1)) = (x, (\partial y) \delta) \end{aligned}$$

where $\delta = \xi_2 - \xi_1$ is a curve around the boundary (one lift of $\partial S_{g,1}$). Now we have

$$\begin{aligned} \langle c, d \rangle_2 - \langle c, d \rangle_1 &= \sum_{h \in H} (c, hd)_2 h - \sum_{h \in H} (c, hd)_1 h \\ &= \sum_{h \in H} (c, \partial(hd) \delta) h \\ &= \bar{\partial} d \sum_{h \in H} (c, h \delta) h \\ &= \bar{\partial} d \langle c, \delta \rangle = \bar{\partial} d \partial c. \end{aligned}$$

In the last line, we used the fact that

$$\begin{aligned} \langle c, \delta \rangle &= \sum_{h \in H} (c, h \delta) h = \sum_{h \in H} (h^{-1} c, \delta) h \\ &= \sum_{h \in H} \text{const}(\partial(h^{-1} c)) h = \partial c, \end{aligned}$$

where by analogy with power series, we use $\text{const}(x)$ to mean the ‘‘constant term’’ of $x \in \mathbb{Z}[H]$; formally, we can define the linear homomorphism $\text{const} : \mathbb{Z}[H] \rightarrow \mathbb{Z}$ that is the identity on $\mathbb{Z} \leq \mathbb{Z}[H]$ and sends h to 0 for all $h \in H \setminus \{0\}$. It now follows that when c or d is a curve, we need only write $\langle c, d \rangle$, since $\langle c, d \rangle_1 = \langle c, d \rangle_2$. \square

We also have the following ‘‘antisymmetry’’ property of $\langle *, * \rangle_i$:

Lemma 4.3. *Suppose $\{i, j\} = \{1, 2\}$ and $c, d \in H_1(\widehat{S}, p^{-1}(b))$. Then $\langle d, c \rangle_i = -\overline{\langle c, d \rangle_j}$. In particular, if c or d is a curve, then $\langle d, c \rangle = -\overline{\langle c, d \rangle}$, so $\langle c, d \rangle = 0 \iff \langle d, c \rangle = 0$.*

Proof. Note that $(f, e)_i = -(e, f)_j$ for any $e, f \in H_1(\widehat{S}, p^{-1}(b))$. Thus

$$\langle d, c \rangle_i = \sum_{h \in H} (d, hc)_i h = - \sum_{h \in H} (hc, d)_j h = - \sum_{h \in H} (c, h^{-1}d)_j h = - \sum_{h \in H} (c, hd)_j h^{-1} = -\overline{\langle c, d \rangle_j},$$

as claimed. \square

Note that this lemma does not imply that $\langle c, c \rangle = 0$ if c is a curve. However, we do have the weaker statement that $\langle c, c \rangle = 0$ if c is a lift of a separating curve in the base surface $S_{g,1}$, as the various lifts $\{hc \mid h \in H\}$ are disjoint in this case.

The following proposition is fundamental for calculations involving the higher intersection form. The result is analogous to the formula for the action of a Dehn twist on H given by $T_\gamma(h) = h + (h, \gamma)\gamma$, where (h, γ) denotes the algebraic intersection number.

Proposition 4.4. *Let γ be a separating curve in $S_{g,1}$ and let $c \in H_1(\widehat{S}, p^{-1}(b))$ be any curve in \widehat{S} lifting γ . Then if T_γ is the Dehn twist around γ , the action of the lifted twist on $H_1(\widehat{S}, p^{-1}(b)) \approx \mathbb{Z}[H]^{2g}$ is*

$$(2) \quad \widetilde{T}_\gamma(d) = d + \langle d, c \rangle c$$

for any $d \in H_1(\widehat{S}, p^{-1}(b))$.

(Recall that a curve γ in $S_{g,1}$ lifts to a curve rather than an arc exactly when γ is a separating curve, so $\langle d, c \rangle = \langle d, c \rangle_1 = \langle d, c \rangle_2$ in the above formula.)

Proof. The lifted homeomorphism \widetilde{T}_γ can be thought of as the simultaneous Dehn twist about all lifts of γ , since these lifts are nonintersecting closed curves in \widehat{S} . For each intersection of d with a lift $\widetilde{\gamma}$ of γ , we add or subtract $\widetilde{\gamma} \in H_1(\widehat{S}, p^{-1}(b))$, depending on the orientation of the intersection. But the lifts of γ are all the curves hc for $h \in H$. Thus

$$\widetilde{T}_\gamma(d) = d + \sum_{\widetilde{\gamma} \text{ lifts } \gamma} (d, \widetilde{\gamma}) \widetilde{\gamma} = d + \sum_{h \in H} (d, hc) hc.$$

From the definition in (1), this is just $\widetilde{T}_\gamma(d) = d + \langle d, c \rangle c$. \square

5. USING THE TRACE OF THE MAGNUS REPRESENTATION

We find it useful to analyze the trace of the Magnus representation r_2 ; this trace gives a class function on $\mathcal{I}_{g,1}$ with values in $\mathbb{Z}[H]$. The formula in Proposition 4.4 gives us a relatively easy way to compute this function on $\mathcal{K}_{g,1}$.

The following statement is equivalent to a theorem of Suzuki ([6], Corollary 4.4); we give a new proof of the latter implication.

Theorem 5.1. *Let γ_1, γ_2 be separating curves in $S_{g,1}$, and c_1, c_2 be lifts to \widehat{S} of γ_1, γ_2 respectively. Then*

$$[T_{\gamma_1}, T_{\gamma_2}] \in \ker r_2 \iff \langle c_1, c_2 \rangle = 0.$$

Proof. First, compute $\widetilde{T}_{\gamma_1} \widetilde{T}_{\gamma_2}$ using (2):

$$(3) \quad \begin{aligned} \widetilde{T}_{\gamma_1} \widetilde{T}_{\gamma_2}(d) &= \widetilde{T}_{\gamma_1}(d + \langle d, c_2 \rangle c_2) \\ &= d + \langle d, c_2 \rangle c_2 + \langle d, c_1 \rangle c_1 + \langle d, c_2 \rangle \langle c_2, c_1 \rangle c_1 \end{aligned}$$

for any $d \in H_1(\widehat{S}, p^{-1}(b))$.

The direction $\langle c_1, c_2 \rangle = 0 \implies [T_{\gamma_1}, T_{\gamma_2}] \in \ker r_2$ follows immediately from (3). In this case the last term vanishes, so the expression becomes symmetric in c_1 and c_2 and \widetilde{T}_{γ_1} and \widetilde{T}_{γ_2} commute in their actions on $H_1(\widehat{S}, p^{-1}(b))$, as desired.

The reverse implication follows from the formula

$$(4) \quad \text{tr}([\widetilde{T}_{\gamma_1}, \widetilde{T}_{\gamma_2}]) = 2g + \langle c_1, c_2 \rangle^2 \langle c_2, c_1 \rangle^2$$

Given this formula, assume $[T_{\gamma_1}, T_{\gamma_2}] \in \ker r_2$. Any $f \in \ker r_2$ must have trace $\text{tr}(\widetilde{f}) = \text{tr}(r_2(f)) = 2g$. Thus $\langle c_1, c_2 \rangle^2 \langle c_2, c_1 \rangle^2 = 0$, implying that $\langle c_1, c_2 \rangle = \langle c_2, c_1 \rangle = 0$ because $\langle c_2, c_1 \rangle = 0 \iff \langle c_1, c_2 \rangle = 0$ by Lemma 4.3. \square

We perform explicitly the simpler computation $\text{tr}(\widetilde{T}_{\gamma_1} \widetilde{T}_{\gamma_2}) = 2g + \langle c_1, c_2 \rangle \langle c_2, c_1 \rangle$ (Theorem 4.4 in [6]) as an example; (4) follows by identical methods.

We have already computed the formula (3) for this linear transformation, and we note that every term is linear in d (recall that $\langle \cdot, \cdot \rangle$ is linear in its first factor). Thus the trace of the whole expression is the sum of the traces of the terms (regarded as linear functions of d):

$$\text{tr}(\widetilde{T}_{\gamma_1} \widetilde{T}_{\gamma_2}) = \text{tr}(d) + \text{tr}(\langle d, c_1 \rangle c_1) + \text{tr}(\langle d, c_2 \rangle c_2) + \text{tr}(\langle d, c_2 \rangle \langle c_2, c_1 \rangle c_1)$$

The first term is the trace of the identity, which has trace $\text{tr}(I_{2g}) = 2g$. For the other terms, recall that in general $\text{tr}(x \mapsto g(x)v) = g(v)$. Thus $\text{tr}(\langle d, c_1 \rangle c_1) = \langle c_1, c_1 \rangle = 0$ (as discussed after Lemma 4.3), and similarly $\text{tr}(\langle d, c_2 \rangle c_2) = \langle c_2, c_2 \rangle = 0$ and $\text{tr}(\langle d, c_2 \rangle \langle c_2, c_1 \rangle c_1) = \langle c_1, c_2 \rangle \langle c_2, c_1 \rangle$. We conclude that

$$\text{tr}(\widetilde{T}_{\gamma_1} \widetilde{T}_{\gamma_2}) = 2g + \langle c_1, c_2 \rangle \langle c_2, c_1 \rangle.$$

In fact, we can show that the commuting relation of Theorem 5.1 is the only relation that ever arises between the images of twists around separating curves.

Theorem 5.2. *Suppose that γ_1 and γ_2 are separating curves on $S_{g,1}$ with lifts $c_1, c_2 \in \mathbb{Z}[H]^{2g}$ such that $\langle c_1, c_2 \rangle \neq 0$. Then there are no relations between $r_2(T_{\gamma_1})$ and $r_2(T_{\gamma_2})$ in $\text{GL}_{2g}(\mathbb{Z}[H])$, i.e. no nontrivial word in T_{γ_1} and T_{γ_2} is in $\ker r_2$.*

Proof. Suppose for contradiction that w is a nontrivial word in T_{γ_1} and T_{γ_2} such that $w \in \ker r_2$. Then without loss of generality, we can assume that w is a word of minimal length in its conjugacy class (as an element of the free group generated by the symbols T_{γ_1} and T_{γ_2}). Then there are two possibilities. First, w might be $T_{\gamma_1}^n$ or $T_{\gamma_2}^n$ for some integer $n \neq 0$. But this is impossible because

$$(5) \quad r_2(T_{\gamma_i}^n)(d) = d + n \langle d, c_i \rangle c_i$$

for all $d \in \mathbb{Z}[H]^{2g}$ by Proposition 4.4, and $\langle \cdot, \cdot \rangle$ is nondegenerate, so $r_2(w) \neq I_{2g}$. In the other possibility, w can be chosen to be of the form

$$w = T_{\gamma_1}^{m_1} T_{\gamma_2}^{n_1} \cdots T_{\gamma_1}^{m_k} T_{\gamma_2}^{n_k},$$

where m_1, \dots, m_k and n_1, \dots, n_k are nonzero integers and $k \geq 1$. Now, expanding $r_2(w)(d)$ via the formula (5) and taking the trace as in the previous theorem, we clearly have that $\text{tr}(r_2(w))$ is a polynomial with integer coefficients in $\langle c_1, c_2 \rangle \langle c_2, c_1 \rangle$ with leading term $m_1 n_1 \cdots m_k n_k (\langle c_1, c_2 \rangle \langle c_2, c_1 \rangle)^k$. But we also know that $\text{tr}(r_2(w)) = \text{tr}(I_{2g}) = 2g$, so $\langle c_1, c_2 \rangle \langle c_2, c_1 \rangle \in \mathbb{Z}[H]$ is the root of a nontrivial polynomial in $\mathbb{Z}[X]$. This implies that $\langle c_1, c_2 \rangle \langle c_2, c_1 \rangle \in \mathbb{Z} \leq \mathbb{Z}[H]$ since H is free abelian, so every element of $\mathbb{Q}[H] \setminus \mathbb{Q}$ is transcendental over \mathbb{Q} . But $\varepsilon(\langle c_1, c_2 \rangle \langle c_2, c_1 \rangle) = (\gamma_1, \gamma_2)(\gamma_2, \gamma_1) = 0$, so we have that $\langle c_1, c_2 \rangle \langle c_2, c_1 \rangle = 0$, which is a contradiction. \square

Now suppose that $\langle c_1, c_2 \rangle = 0$. Then by Theorem 5.1 we know that $r_2(T_{\gamma_1})$ and $r_2(T_{\gamma_2})$ must commute. But this is again the only relation that arises.

Theorem 5.3. *Suppose that γ_1 and γ_2 are nonisotopic separating curves on $S_{g,1}$ with lifts c_1, c_2 such that $\langle c_1, c_2 \rangle = 0$. Then the only relations between $r_2(T_{\gamma_1})$ and $r_2(T_{\gamma_2})$ in $GL_{2g}(\mathbb{Z}[H])$ are “commuting” relations; $r_2(T_{\gamma_1})$ and $r_2(T_{\gamma_2})$ generate a free abelian group of rank 2.*

Proof. This is a simple consequence of the fact that $r_2(T_{\gamma_1}^m T_{\gamma_2}^n)(d) = d + m\langle d, c_1 \rangle c_1 + n\langle d, c_2 \rangle c_2$. The lifts of two nonisotopic separating curves are linearly independent, so nondegeneracy of $\langle \cdot, \cdot \rangle$ implies that $T_{\gamma_1}^m T_{\gamma_2}^n \in \ker r_2$ only if $m = n = 0$, as desired. \square

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