

CORNELL UNIVERSITY MATHEMATICS DEPARTMENT SENIOR THESIS

***Properties of Posets in Non-crossing  
Pairings on Bitstrings***

A THESIS PRESENTED IN PARTIAL FULFILLMENT  
OF CRITERIA FOR HONORS IN MATHEMATICS

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ABSTRACT. There is a standard bijection between the non-crossing partition lattice  $NC(n)$  and the set of non-crossing pairings  $NC_2(2n)$  in the larger lattice  $NC(2n)$ . We study the image of this bijection on subsets of  $NC_2(2n)$  corresponding to bit strings of length  $2n$ . We show that the resulting posets are connected, convex sub-posets of  $NC(n)$ . We give simple algorithms to calculate the extremal elements in these posets. We show that each set of partitions mapped from subsets of  $NC_2(2n)$  corresponding to bit strings of length  $2n$  is unique up to inversion of 1s and 0s. Finally, we show that the resulting poset is constructible.

## 1. INTRODUCTION

**1.1. Posets of Non-Crossing Partitions.** We follow the introduction written in [1] by Kemp as background to this thesis.

Let  $[n]$  denote the set  $\{1, 2, \dots, n\}$ , and let  $\mathcal{P}(n)$  denote the set of all partitions of  $[n]$ ; that is,  $\pi \in \mathcal{P}(n)$  means that  $\pi$  is a collection  $\{B_1, \dots, B_r\}$  of non-empty disjoint subsets  $B_j \subseteq [n]$  such that  $B_1 \cup \dots \cup B_r = [n]$ . (The subsets  $B_j$  are called the *blocks* of  $\pi$ .) There is a natural poset ordering on  $\mathcal{P}(n)$ : namely, we say  $\pi \leq \sigma$  if  $\pi$  is a *refinement* of  $\sigma$  (that is, each block of  $\sigma$  is contained in a block of  $\pi$ ). It is a classical result (see, for instance [7], page 127) that  $(\mathcal{P}(n), \leq)$  is actually a graded lattice, graded by the number of blocks in each partition; the maximal and minimal elements (usually denoted  $0_n$  and  $1_n$ ) are pictured below.

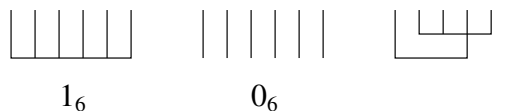


FIGURE 1. The maximal and minimal elements,  $1_6$  and  $0_6$ , in the lattice  $\mathcal{P}(6)$ , along with another (crossing) partition of  $[6]$ . They are represented here as linear partition diagrams, which we will use through much of this thesis.

There is a natural linear order on  $[n]$ , and from it we can define *crossings*: given a partition  $\pi$  of  $[n]$ , two blocks  $B_1, B_2 \in \pi$  are said to *cross* if there exist  $i_1, j_1 \in B_1$  and  $i_2, j_2 \in B_2$  such that  $i_1 < i_2 < j_1 < j_2$ . An example of a partition with a crossing is also given in Figure 1. A partition is called *non-crossing* if no two of its blocks cross. The set of non-crossing partitions of  $[n]$  is denoted  $NC(n)$ , and is a graded lattice in its own right under the refinement ordering. It is, in fact, more symmetric than  $\mathcal{P}(n)$ , as it is self-dual (we will discuss the associated complementation map below).

Both  $\mathcal{P}(n)$  and  $NC(n)$  were explicitly enumerated long ago;  $|\mathcal{P}(n)|$  is the *Bell number*  $B_n$ , while  $|NC(n)|$  is counted by the *Catalan number*  $C_n = \frac{1}{n+1} \binom{2n}{n}$ . This last result may be proved recursively quite easily, but for our purposes there is a somewhat simpler proof that goes through the *non-crossing pairings*. A *pairing* is a partition each of whose blocks has two elements. Evidently, there are no pairings of  $[n]$  if  $n$  is odd;  $\mathcal{P}_2(2n)$  denotes the set of pairings of  $[2n]$  and  $NC_2(2n)$  denotes the non-crossing pairings of  $[2n]$ . It is easy to check that  $|\mathcal{P}_2(2n)| = (2n - 1)!!$ . It turns out that  $|NC_2(2n)|$  is also counted by the Catalan number  $C_n$ , as Figure 2 demonstrates. This result affords a slightly different proof that  $|NC(n)| = C_n$ , which motivates our main results that follow. We record this proof in Figure 2.

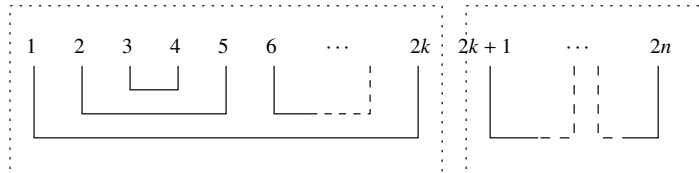


FIGURE 2. In any non-crossing pairing of  $[2n]$ ,  $1$  must pair to an even number (else there would be an odd number of indices enclosed by the pairing, forcing a crossing). Hence,  $1$  can pair to any of  $2, 4, \dots, 2n$ . If it pairs to  $2k$  then, as the figure demonstrates, the non-crossing condition factors the remaining possible pairings into two independent factors: the pairings of the  $2(k - 1)$  indices enclosed by the block  $\{1, 2k\}$ , and the remaining  $2(n - k)$  indices to the right. So, conditioned on  $1$  pairing to  $2k$ , there are  $|NC_2(2(k - 1)) \cdot NC_2(2(n - k))|$  pairings. In total, then,  $NC_2(2n) = \sum_{k=1}^n |NC_2(2(k - 1))| \cdot |NC_2(2(n - k))|$ ; this is the recurrence relation for the Catalan numbers.

**Proposition 1.1.** *There is a bijection  $\theta_n: NC_2(2n) \rightarrow NC(n)$ .*

*Proof.* The idea is to map  $[2n]$  onto  $[n]$  in a  $2-1$  fashion, such that the induced map on partitions preserves the non-crossing conditions. There are many ways to do this; we will use the map  $h(j) = \lceil j/2 \rceil$ . For notational clarity we will use boldface when referring to the elements of  $[n]$ . Explicitly, given any partition  $\pi \in \mathcal{P}(2n)$ , we form the partition  $\theta_n(\pi) \in \mathcal{P}(n)$  as follows:  $\mathbf{i}, \mathbf{j} \in [n]$  are in the same block of  $\theta_n(\pi)$  iff the sets  $h^{-1}\{\mathbf{i}\} = \{2i-1, 2i\}$ ,  $h^{-1}\{\mathbf{j}\} = \{2j-1, 2j\}$  share a common block. Figure 3 below demonstrates.

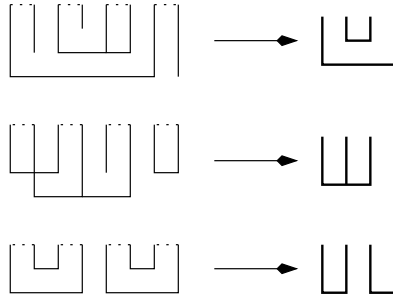


FIGURE 3. Three partitions in  $\mathcal{P}(8)$ , and their images under  $\theta_4$ . The action of  $h$  is to collapse each consecutive pair  $\{2j-1, 2j\}$  to the single index  $\mathbf{j}$ , hence we can visualize the action of  $\theta_n$  by connecting these neighbours (as we have done with dotted lines above).

Now, let  $\pi \in NC(2n)$ . Suppose (for a contradiction) that  $\theta_n(\pi)$  has a crossing – say  $B_1, B_2 \in \theta_n(\pi)$  cross:  $\mathbf{i}_1 < \mathbf{i}_2 < \mathbf{j}_1 < \mathbf{j}_2$ , where  $\mathbf{i}_1, \mathbf{j}_1 \in B_1$  and  $\mathbf{i}_2, \mathbf{j}_2 \in B_2$ . By definition,  $\mathbf{i}_1, \mathbf{j}_1 \in B_1 \in \theta_n(\pi)$  means that one of the two numbers  $h^{-1}\{\mathbf{i}_1\} = \{2i_1-1, 2i_1\}$  shares a common block  $V_1 \in \pi$  with one of the two numbers  $h^{-1}\{\mathbf{j}_1\} = \{2j_1-1, 2j_1\}$ . Similarly, one of the two numbers  $h^{-1}\{\mathbf{i}_2\} = \{2i_2-1, 2i_2\}$  shares a common block  $V_2 \in \pi$  with one of the two numbers  $h^{-1}\{\mathbf{j}_2\} = \{2j_2-1, 2j_2\}$ . Notice that  $2i_1-1 < 2i_1 < 2i_2-1 < 2i_2 < 2j_1-1 < 2j_1 < 2j_2-1 < 2j_2$ . It is easy to see, therefore, that the blocks  $V_1, V_2 \in \pi$  must cross, which is a contradiction. Hence,  $\theta_n$  maps  $NC(2n)$  into  $NC(n)$ .

In particular, the image  $\theta_n(NC_2(2n))$  of  $\theta_n$  on non-crossing pairings is contained in  $NC(n)$ . We will now show that this mapping is a bijection, by explicitly producing its inverse. From Figure 3 above, we see that  $\theta_n(\pi)$  is produced from  $\pi$  by “collapsing” each consecutive pair of indices in  $[2n]$  into a single index in  $[n]$ . To reverse

this, we must “double” each line in the partition diagram of the target image. Let  $\sigma \in NC(n)$ , and consider any block  $B = \{\mathbf{i}_1, \dots, \mathbf{i}_k\}$  in  $\sigma$ , where  $\mathbf{i}_1 < \dots < \mathbf{i}_k$ . Doubling these lines simply means that we must include the following 2-blocks in  $\varphi_n(\sigma)$ :  $\{2i_1 - 1, 2i_k\}, \{2i_1, 2i_2 - 1\}, \{2i_2, 2i_3 - 1\}, \dots, \{2i_{k-1}, 2i_k - 1\}$ . An argument much like the one above shows that the pairing  $\varphi_n(\sigma)$  is non-crossing, and it is a simple matter to check that  $\varphi_n: NC(n) \rightarrow NC_2(2n)$  and  $\theta_n: NC_2(2n) \rightarrow NC(n)$  are inverses.  $\square$

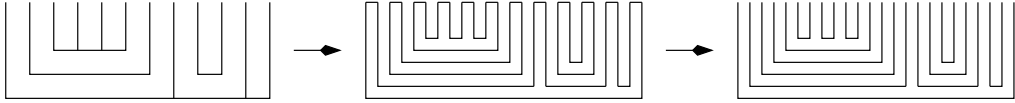


FIGURE 4. The action of  $\varphi_{12}$  on a partition  $\pi$  in  $NC(12)$ . Topologically, we take a tubular neighbourhood of the 1-skeleton that represents the partition diagram of  $\pi$ , and then take the boundary of this minus the top boundary segments. The resulting 1-skeleton is the partition diagram of  $\varphi_{12}(\pi)$  in  $NC_2(24)$ .

*Remark 1.2.* Aside from giving a concrete bijection between  $NC_2(2n)$  and  $NC(n)$  for enumeration purposes, the map  $\theta_n$  allows us to import the lattice structure of  $NC(n)$  into  $NC_2(2n)$ . Note that every pairing in  $NC_2(2n)$  has  $n$  blocks, and so all pairings are incomparable in the refinement order on  $NC(2n)$ . But we can pull back the ordering of  $NC(n)$  via  $\theta_n$ , as follows.

**Definition 1.3.** Given  $\pi, \sigma \in NC_2(2n)$ , say that  $\pi \leq_{\theta} \sigma$  iff  $\theta_n(\pi) \leq \theta_n(\sigma)$  in  $NC(n)$ . Since  $\theta_n$  is a bijection, this means that  $(NC_2(2n), \leq_{\theta})$  is a lattice isomorphic to  $NC(n)$ . Refer to  $\leq_{\theta}$  as the **thickened ordering** on  $NC_2(2n)$ .

Hence, any subset  $T \subseteq NC_2(2n)$  has a natural poset structure induced by the thickened ordering on  $NC_2(2n)$ . This brings us to the special subsets of non-crossing pairings we are motivated to study in this thesis.

**1.2. Knights and Ladies of the Round Table.** King Arthur is hosting a dinner for his  $n$  Knights, to which he has invited  $n$  Ladies. The  $2n$  guests seat themselves randomly around the Round Table, before Arthur can assign seats to them. Given their seating arrangement, in how many distinct ways can the Knights and Ladies pair off to chat without any conversations crossing?

Put in slightly more precise terms, the pairings in question are defined as follows.

**Definition 1.4.** Let  $\mathbf{s} = s_1 \cdots s_{2n}$  be a bit string that is balanced:  $|\{j; s_j = 0\}| = |\{j; s_j = 1\}| = n$ . The set of **non-crossing pairings of  $\mathbf{s}$** , denoted  $NC_2(\mathbf{s})$ , is the set of non-crossing pairings  $\pi \in NC_2(2n)$  with the property that, for any block  $B = \{i, j\} \in \pi$ ,  $s_i \neq s_j$ . That is,  $NC_2(\mathbf{s})$  consists of those non-crossing pairings that match 1s to 0s in  $\mathbf{s}$ .

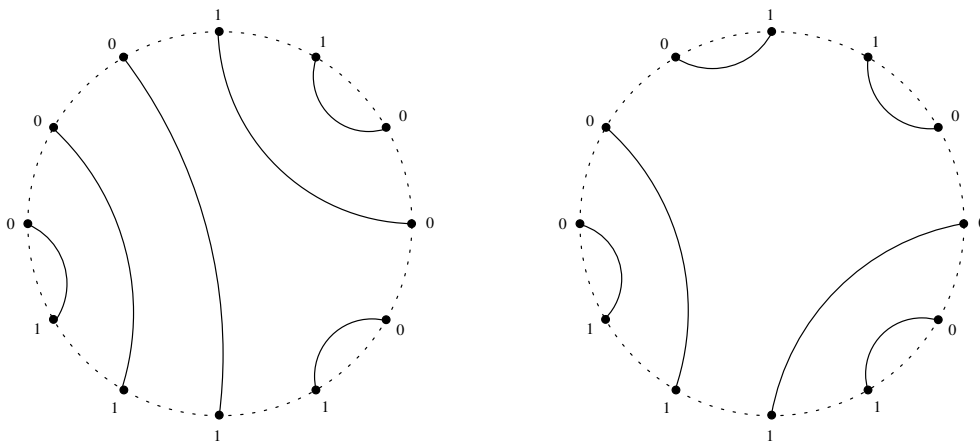


FIGURE 5. Two non-crossing pairings of the bit string  $\mathbf{s} = 110001111000$ . They are represented here in circular form.

*Remark 1.5.* Cyclic permutations of  $[n]$  induce lattice isomorphisms of  $NC(n)$  (and these, together with the reflection  $[n] \mapsto [n]^* = (n, n-1, \dots, 2, 1)$ , generate all lattice isomorphisms, see [6]). Hence, it is possible to draw partition diagrams either on a line as we have been doing up to now, or on a circle as in Figure 5 above. However, as we will discuss in Section 3, the thickened ordering  $\leq_\theta$  on  $NC_2(2n)$  is *not* preserved under all cyclic permutations of  $[2n]$ , and so we typically prefer the linear representation when convenient.

The problem of the Knights and Ladies of the Round Table is the question of enumerating  $NC_2(\mathbf{s})$  for any given balanced bit string  $\mathbf{s}$ . This is a surprisingly difficult problem, which has been addressed in different ways in the recent papers [2, 3, 4, 5]. This set of pairings has important connections with random matrix theory and free probability theory, which provided the motivation for its study. In this thesis, we are primarily interested in studying the poset structure  $(NC_2(\mathbf{s}), \leq_\theta)$ . To that end, we introduce some useful notation.

**Notation 1.6.** For a balanced bit string  $\mathbf{s} \in \{0, 1\}^{2n}$ , let  $T_{\mathbf{s}} = T(\mathbf{s})$  denote the sub-poset  $\theta_n(NC_2(\mathbf{s}))$  of  $NC(n)$ . Alternatively,  $T_{\mathbf{s}}$  denotes the set  $NC_2(\mathbf{s})$  equipped with the thickened ordering  $\leq_{\theta}$ .

*Example 1.7.* In any non-crossing pairing of  $[2n]$ , each block must pair two indices of opposite parity, or else there would be an odd number of indices enclosed by that block resulting in a crossing. In other words, any non-crossing pairing is automatically in  $NC_2(1010 \cdots 10)$ . Thus  $T((10)^n)$  is equal (as a poset) to  $NC(n)$ .

*Example 1.8.* The reader may readily verify that the bit string  $1^n 0^n$  has only one pairing, pictured in Figure 6 below. Hence,  $T(1^n 0^n)$  is the trivial singleton poset.

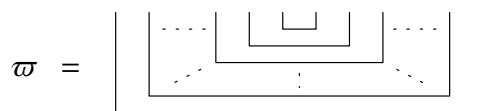


FIGURE 6. The single pairing  $\varpi$  of  $1^n 0^n$ .

*Example 1.9.* Consider the bit string  $\mathbf{s} = 1011010010$ . The set  $NC_2(\mathbf{s})$  contains 10 pairings. These pairings, their image under  $\theta_5$  in  $NC(5)$ , and the poset  $T_{\mathbf{s}}$ , are displaying in Figure 7.

## 2. THE KREWERAS COMPLEMENT

Following the above background, we will now prove a few theorems classifying the structure of all the posets  $T_{\mathbf{s}}$ . In order to do so, we first define the Kreweras complement. Let  $\{1, 2, \dots, n\}$  and  $\{1', 2', \dots, n'\}$  be two disjoint sets of  $n$  elements, and order the union of these two sets  $\{1 < 1' < 2 < 2' < \dots < n < n'\}$ .

**Definition 2.1.** The Kreweras complement of  $\pi \in NC(1, 2, \dots, n)$  is the largest element  $\sigma \in NC(1', 2', \dots, n')$  such that  $\pi \cup \sigma \in NC(1, 1', 2, 2', \dots, n, n')$  [6].

Rotation of  $\pi \in NC_2(2n)$  by an even number of places corresponds to a rotation of  $\theta_n(\pi)$ , since we are essentially rotating the  $n$  underlying elements of the partition  $\theta_n(\pi)$ . Rotation by one element to the left, shown in Figure 8, gives the Kreweras complement.

**Lemma 2.2.** Let  $\pi'$  be  $\pi$  rotated one element to the left. Then,  $\theta_n(\pi')$  is the Kreweras complement of  $\theta_n(\pi)$ .



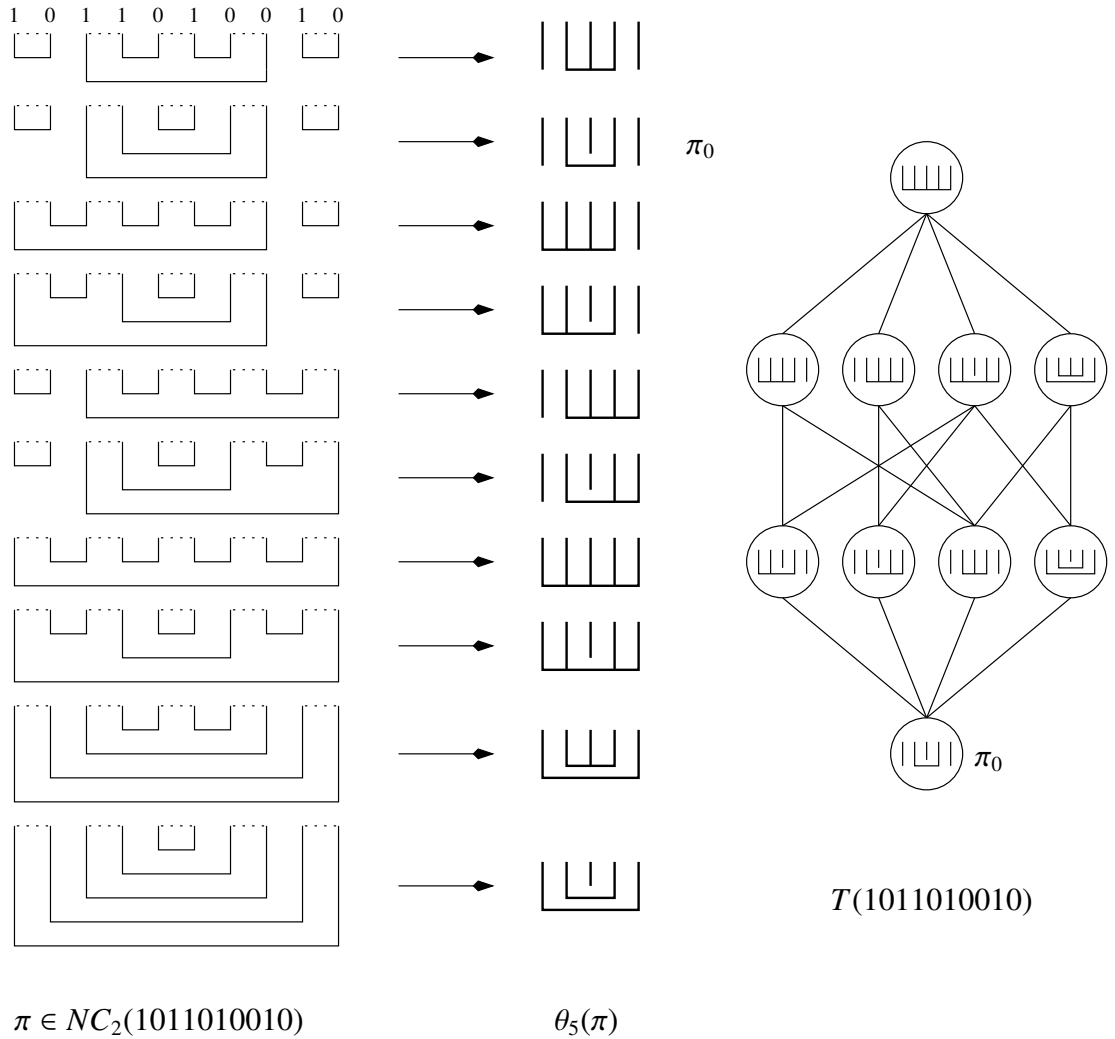


FIGURE 7. The ten pairings in  $NC_2(s)$  with  $s = 1011010010$ , and their images under the thickening  $\theta_5$ . Also shown is the poset  $T_s$  in  $NC(5)$ . The astute observer will notice that  $T_s$  is actually equal to the whole interval  $[\pi_0, 1_5]$  in  $NC(5)$ .

*Proof.* The function  $\theta_n$  acting on  $\pi'$  joins the  $(2i)^{th}$  and  $(2i+1)^{th}$  elements of the original  $\pi$  for  $i \geq 1$ , and brings the first element to the right and joins it with the  $(2n)^{th}$  element, with blocks as described earlier. Now, if we place these  $n$  elements between our original  $n$  elements in the way described in the definition of the Kreweras complement,  $\theta_n(\pi')$  is a partition in  $NC(n)$  whose union with the original partition  $\theta_n(\pi)$  is in  $NC(2n)$ . To see this, note that in order for a crossing to occur, there

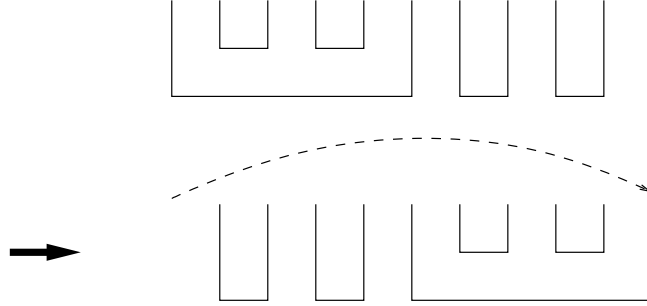


FIGURE 8.

must be some elements  $\mathbf{i} < \mathbf{j}$  of  $\theta_n(\pi)$  and  $\mathbf{i}' < \mathbf{j}'$  of  $\theta_n(\pi')$  such that  $\mathbf{i}$  and  $\mathbf{j}$  are in a block  $a$ , and  $\mathbf{i}'$  and  $\mathbf{j}'$  are in a block  $b$  with  $\mathbf{i} < \mathbf{i}' < \mathbf{j} < \mathbf{j}'$ . Furthermore, these elements can be chosen so that

- There are no elements of either  $a$  or  $b$  between  $\mathbf{i}'$  and  $\mathbf{j}$ .
- $\mathbf{i}$  is the greatest element of  $a$  which is less than  $\mathbf{i}'$ .
- $\mathbf{j}'$  is the least element of  $b$  which is greater than  $\mathbf{j}$ .

But this would imply that there is a pairing between element  $2i'$  and  $2j'-1$  in  $\pi'$ , and another between  $2j-1$  and  $2i$  in  $\pi$ , which corresponds to crossing pairings in  $\pi$ .

Now we show that  $\theta_n(\pi')$  is the largest partition whose union with  $\theta_n(\pi)$  is in  $NC(2n)$ , which tells us that  $\theta_n(\pi')$  is the Kreweras complement of  $\theta_n(\pi)$ . We prove this by contradiction. If such a partition  $\sigma$  is strictly larger than  $\theta_n(\pi')$ , then at least one of its blocks contains more than one block of  $\theta_n(\pi')$ . Since the union of  $\sigma$  with  $\theta_n(\pi)$  is non-crossing, any refinement in the interval  $[\theta_n(\pi'), \sigma]$  has a union with  $\theta_n(\pi)$  that is non-crossing, so we can take two blocks of  $\theta_n(\pi')$  that lie in the same block of  $\sigma$  and join them without introducing any crossings. Let  $a$  and  $b$  be two such blocks. Find a pair of elements,  $\mathbf{i}'$  from  $a$  and  $\mathbf{j}'$  from the  $b$ , such that no elements from either block are in between those. Without loss of generality, assume that  $\mathbf{i}' < \mathbf{j}'$ . Consider the underlying pairing of  $\pi'$  where the bits  $2i' - 1$  and  $2i'$  correspond to element  $\mathbf{i}'$  in  $\theta_n(\pi')$ . Note that these are really the bits  $2i$  and  $2i + 1$  in  $\pi$ . We see that  $2i + 1$  is paired to some bit  $2k$ . Then in  $\theta_n(\pi)$ , the element  $\mathbf{i} + 1$  is in the same block as the element  $\mathbf{k}$ . Since the union of  $\theta_n(\pi')$  and  $\theta_n(\pi)$  is non-crossing,  $\mathbf{k}$  is neither to the left of  $\mathbf{i}$  nor to the right of  $\mathbf{j}$ . Thus, in  $\theta_n(\pi)$ ,  $\mathbf{i} < \mathbf{k} < \mathbf{j}$ . But then  $a$  has an element between  $\mathbf{i}$  and  $\mathbf{j}$ , a contradiction. Hence, rotation by an odd number of places gives us a rotation of the Kreweras complement of the partition.  $\square$

## 3. CONVEXITY

Our next goal is to show that  $T_s$  is convex. A subset  $Q$  of a poset  $P$  is *convex* if whenever  $a$  and  $c$  are in  $Q$ , and  $a \leq b \leq c$ , then  $b \in Q$ .

**Theorem 3.1.** *Let  $s$  be a balanced bit string in  $\{0, 1\}^{2^n}$ . Then the poset  $T_s$  is convex in  $NC(n)$ .*

Before we prove this we require some preliminary results and definitions.

**Definition 3.2.** *Given a bitstring  $s$  in  $\Omega^n$ , a partition in  $NC(n)$  is  $s$ -valid if it is in  $T_s$ .*

**Definition 3.3.** *A block  $A$  is nested if there is some block which has an element to the left of the first element of  $A$  and an element to the right of the last element of  $A$ . A block which is not nested is top-level.*

Let  $a$  and  $c$  be  $s$ -valid with  $a \leq c$ . Pick a block  $X$  of  $c$ . Consider any  $b$  in  $[a, c]$ . Let  $b'$  be  $b$  restricted to  $X$ ,  $a'$  be  $a$  restricted to  $X$ ,  $c'$  be  $c$  restricted to  $X$  (which is  $1_X$ ), and  $s'$  be the substring associated with  $X$ . Pick a block  $Y$  of  $b'$ .

**Lemma 3.4.** *If there are two top-level blocks in  $a'$  restricted to  $Y$ , then we can merge them to form a new partition  $a_1$  which is  $s$ -valid.*

*Proof.* Let the two blocks be as shown below.

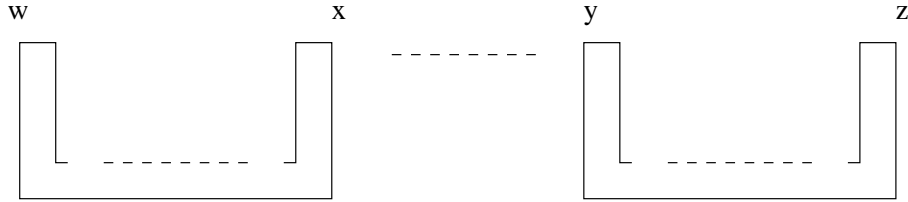


FIGURE 9.

Let  $w$ ,  $x$ ,  $y$ , and  $z$  be as in Figure 9. Since the blocks are part of  $Y$ , any blocks with elements between the elements  $x$  and  $y$  in  $b'$  are nested in  $Y$ , and since  $a'$  is a refinement of  $b'$ , any block of  $a'$  with an element between the two blocks shown only has elements between those two blocks. So the elements in  $s'$  between  $x$  and  $y$  are paired only amongst themselves. Suppose  $w$  is 1. Then  $x$  is 0, since  $a'$  is  $s'$ -valid and  $w$  is paired to  $x$  in  $a'$ . Let the element to the right of  $x$  be  $u_1$ . Since  $c$  restricted to  $X$  is the maximal element, every even-numbered element is paired to the next element (which is odd-numbered). So

$u_1$  is 1. We know  $u_1$  is paired with some element  $v_1$  in  $a'$  before  $y$ , so  $v_1$  is 0, and it is in turn paired to the element to its right  $u_2$  in  $c'$ , and so on. Since there are a finite number of elements between  $x$  and  $y$ , there will be some  $u_l = y$ , whereupon this process terminates. Because  $y$  is paired to  $z$  in  $a'$ ,  $z$  is 0. This process is shown in Figure 10.

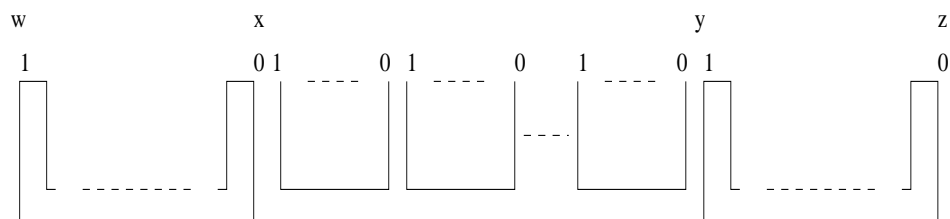


FIGURE 10.

Now we can make a different pairing of  $s$  with  $w$  paired to  $z$  and  $x$  paired to  $y$ , and everything else paired as in  $a$ . This is non crossing, as elements between  $x$  and  $y$  only pair between themselves, so the partition we get from this new pairing is  $s$ -valid, and this partition is  $a$  with those two blocks joined, as shown in Figure 11.  $\square$

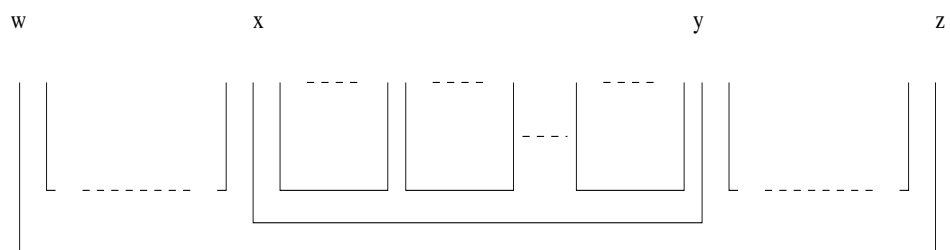


FIGURE 11.

**Lemma 3.5.** *If there are blocks in  $a'$  restricted to  $Y$  nested in a top-level block of  $a'$  restricted to  $Y$ , we can merge one of the nested blocks with the top level block to form a new partition  $a_1$  which is  $s$ -valid.*

*Proof.* Picking such a top-level block of  $a'$  restricted to  $Y$ , choose the leftmost such nested block. These blocks are as shown in Figure 12.

Following a similar argument as in 3.4, the elements between  $w$  and  $x$  are paired only to themselves, and similarly for the elements between  $y$  and  $z$ . Then, continuing that argument along the same

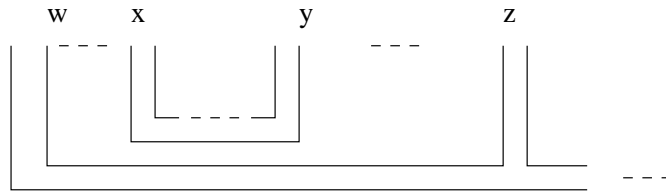


FIGURE 12.

lines, if  $w$  is 1, then  $x$  is 0,  $y$  is 1, and  $z$  is 0, so we can pair  $w$  to  $x$  and  $y$  to  $z$  instead, giving a new partition  $a_1$  which is  $s$ -valid, joining the two blocks as in Figure 13.  $\square$

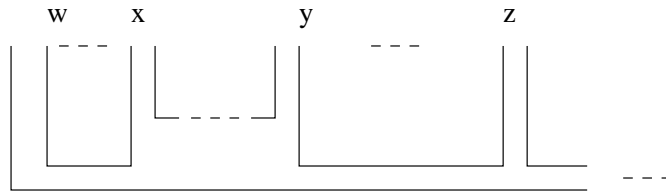


FIGURE 13.

Now we are ready to prove Theorem 3.1.

*Proof.* Consider any  $b$  in  $[a, c]$ . Let the blocks of  $c$  be  $X_1, X_2, \dots, X_k$ . If for every block  $X_i$  of  $c$ ,  $b$  restricted to  $X_i$  is valid with respect to the substring associated with that block  $X_i$ , then  $b$  is  $s$ -valid, since we can then pair the elements in each block to form a pairing which maps to  $b$ . Thus we only need to consider a particular block  $X$  of  $c$ . Note that  $c$  restricted to  $X$  is the maximal element. Let  $b', a', c'$ , and  $s'$  be as in the lemmas above. If  $b'$  is the maximal element or  $a' = b'$ , then  $b'$  is  $s'$ -valid, and we are done. If not, we consider the blocks  $Y_1, Y_2, \dots, Y_k$  of  $b'$ . There is some block  $Y$  of  $b'$  such that  $a'$  restricted to  $Y$  is not the maximal element. Now we can apply either of the two lemmas to  $a$  to get  $a_1$  such that  $a_1$  is  $s$ -valid, and  $a < a_1 \leq b \leq c$ . Repeat the same process to  $a_1$  to get  $a_2$ , and so on. Since the number of blocks of  $a$  and  $b$  are finite, we will eventually get some  $a_m$  such that  $a_m$  has the same number of blocks as  $b$ , so  $a_m = b$ , and  $a_m$  is  $s$ -valid. Thus  $b$  is  $s$ -valid as well.  $\square$

It follows from Theorem 3.1 that, if  $T_s$  ( $s \in \{0, 1\}^{2n}$ ) has a unique maximal element  $M$  and minimal element  $m$ , then  $T_s = [m, M]$  consists of the interval in  $NC(n)$  between them; an example is demonstrated in Figure 7. In general, as we will see,  $T_s$  may have several maxima  $M_1, \dots, M_r$  and minima  $m_1, \dots, m_s$ . Since  $T_s$  is convex,

$$T_s = \bigcup_{\substack{1 \leq i \leq r \\ 1 \leq j \leq s}} [m_i, M_j].$$

So  $T_s$  is completely determined by identifying its maxima and minima. In order to understand these, we introduce the following.

**Definition 3.6.** Let  $s$  be a balanced bit string in  $\{0, 1\}^{2n}$ . Construct a string  $\delta(s)$  in  $\{\diamond, 0, 1\}^n$  as follows. For each  $j \in [n]$ , if  $s_{2j-1} = s_{2j}$  then  $\delta(s)_j = s_{2j}$ . Otherwise, if  $s_{2j-1} \neq s_{2j}$ , then  $\delta(s)_j = \diamond$ .

*Example 3.7.* Let  $s = 1011010010$  as in Example 1.9. Then  $\delta(s) = \diamond 1 \diamond 0 \diamond$ . On the other hand, with  $t = 111000111000$ , we have  $\delta(t) = 1 \diamond 0 1 \diamond 0$ .

The construction insures that the (ordered) set of 1s and 0s in  $\delta(s)$  forms a balanced substring: the  $\diamond$ s correspond to 01 (or 10) pairs in  $s$ , so removing them means removing equal numbers of 1s and 0s. Hence, since  $s$  is balanced, the remaining substring is also balanced.

**Notation 3.8.** Let  $\delta \in \{\diamond, 1, 0\}^n$ , with balanced  $\{0, 1\}$ -substring. Denote by  $NC_{1,2}(\delta)$  the set of partitions  $\pi \in [n]$  such that:

- If  $\delta_j = \diamond$  then  $\{j\}$  is a singleton block in  $\pi$ .
- If  $\delta_j \in \{0, 1\}$ , then  $j$  is in a two-block  $\{j, k\} \in \pi$  where  $\delta_k = 1 - \delta_j$ .

That is,  $\pi$  contains only one-blocks and two-blocks, where the one-blocks are the  $\diamond$ s in  $\delta$  and the two blocks pair 1s with 0s in  $\delta$ .

This allows us to easily describe the minimal elements in  $T_s$ .

**Theorem 3.9.** Let  $s$  be a balanced bit string in  $\{0, 1\}^{2n}$ . Then the minimal elements of the poset  $T_s \subseteq NC(n)$  are precisely the partitions in  $NC_{1,2}(\delta(s))$ .

*Proof.* First, we note that a minima of this poset must have block sizes of either 1 or 2. If it has a block with size greater than 2, then the substring associated with it must have only one pairing, that which gives the maximal element. If not, there would be another pairing of that substring which corresponds to a valid refinement of the partition, contradicting minimality. But the only strings with just one pairing are rotations of  $11\dots 100\dots 0$ , and the partition associated with such pairings is not the maximal element in  $NC(i)$  for  $i > 2$ . Thus the blocks all have size at most 2. In fact, this shows that the associated

substring for a 2-block is 1100 or 0011, and that for a 1-block is 10 or 01.

This means that if two elements of the string,  $s_{2j-1}$  and  $s_{2j}$ , are different, then they are paired together and form a 1-block in any minima. If they are of the same type, they are paired to two elements  $s_{2k-1}$  and  $s_{2k}$ , of a different type,  $s_{2j-1}$  with  $s_{2k}$ , and  $s_{2k-1}$  and  $s_{2j}$ . This forms a 2-block in the minima. This gives a simple algorithm to determine all possible minima. First we determine all the singletons, which are those pairs  $s_{2j-1} \neq s_{2j}$ . The remaining pairs of elements  $s_{2j-1}$  and  $s_{2j}$  that have the same value are paired with another pair of elements as described above, forming a 2-block. This corresponds to a pairing of the associated bits in  $\delta(s)$ . The set of all such partitions is precisely  $NC_{1,2}(\delta(s))$ . Hence any minima must be an element of  $NC_{1,2}(\delta(s))$ .

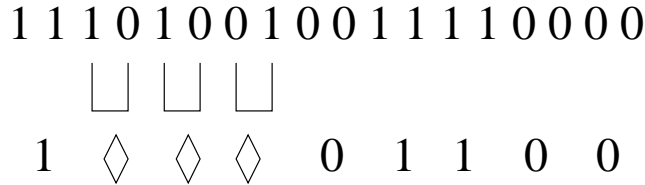


FIGURE 14.

Any pairing of  $\delta(s)$  corresponds to a pairing of the elements of the original string by inverting the above process. One such example is shown in Figure 14 and Figure 15.

These are all indeed  $s$ -valid, and cannot be refined further, so they are all minima. Since all minima must have this form, the minima are exactly the elements of  $NC_{1,2}(\delta(s))$ . Note that all of them have the same number of blocks (the number of singletons plus the number of pairings).  $\square$

Because the Kreweras complementation map reverses order, we can also obtain the maxima via the same method. It follows that all of the maxima have the same number of blocks. To do this, rotate the string by moving the leftmost element to the right as shown in Figure 16. Perform the procedure as above, then shift back the added lines, giving us the maxima (Figure 17).

The number of 2-blocks in the minima is the number of 1s in  $\delta(s)$ , which is the number  $n_1$  of adjacent 1s with the first in an odd position and the second in an even position. Thus the number of blocks is  $n - n_1$ . Similarly, the number of blocks in each maxima is  $n + 1 -$

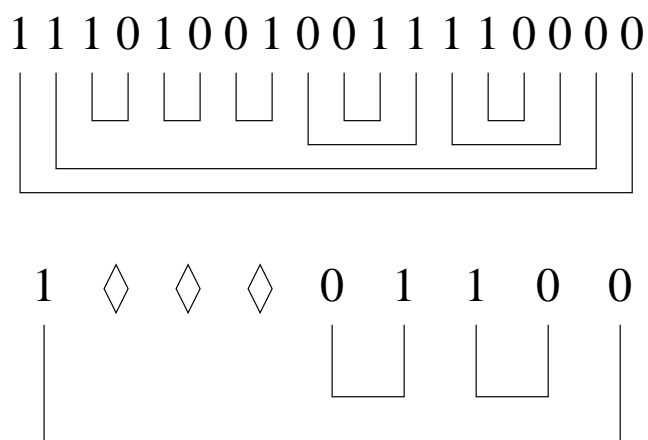


FIGURE 15.

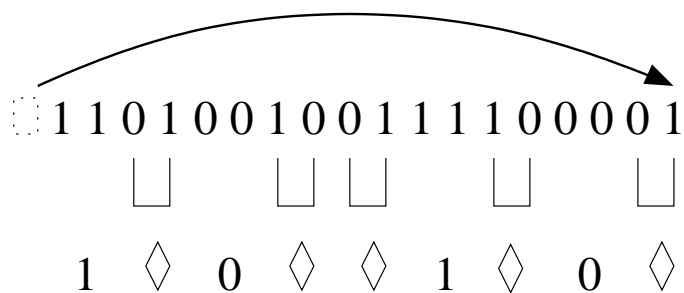


FIGURE 16.

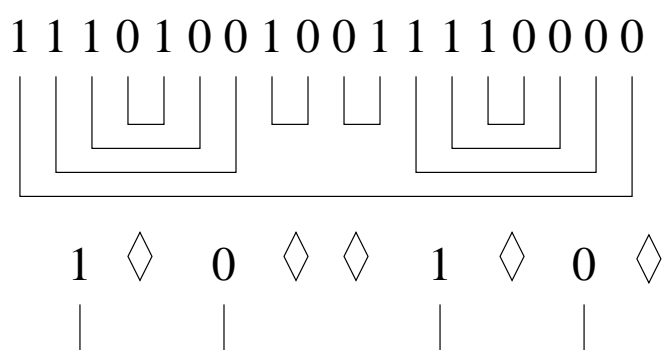


FIGURE 17.

$(n - n_2) = 1 + n_2$ , where  $n_2$  is the number of adjacent 1s with the first in an even position and the second in an odd position, taking the first and last bits to be adjacent. Thus the maximum chain length is



$n - n_1 - 1 - n_2$  which is  $n - 1$  minus the total number of pairs of adjacent 1s. Note that each sequence of  $k$  adjacent 1s bordered on each side by a 0 contributes  $k$  to the total count of 1s, and  $k - 1$  to the number of adjacent pairs of 1s. Then if there are in total  $m$  such sequences, there are  $n - m$  pairs of adjacent 1s, so the maximum chain length is  $m - 1$ . This give us the following proposition.

**Proposition 3.10.** *If there are  $m$  distinct sequences of adjacent 1s bordered by 0s on either side in  $s$ , taking the first and last bit as adjacent, then  $T_s$  is graded with length  $m - 1$ .*

We can use this to easily determine the structure of all such  $T_s$  graded with length 1. First, we present a lemma.

**Lemma 3.11.** *In  $T_s$ , any element 1 rank above the minima is greater than not more than 2 minima, and any element 1 rank below the maxima is less than not more than 2 maxima.*

*Proof.* An element  $a$  1 rank above the minima has the same structure as a minimum less than it, except that two of its blocks are joined together into one block. None of its other blocks can be separated to form a minimum, since its other 2-blocks cannot be broken up into 1-blocks to form a minimum from Theorem 3.9. If two 1-blocks were joined, the resulting block can only be partitioned in one way to form a minimum. Likewise if a 1-block and 2-block were joined, since the 1-block must be present in any minimum, from Theorem 3.9. If two 2-blocks were joined, they must be partitioned into two 2-blocks by Theorem 3.9, and a 4-block can be partitioned into two noncrossing 2-blocks in only two ways. Thus there are at most two minima below  $a$ . For an element 1 rank below the maxima, rotating one bit inverts the poset, so it becomes 1 rank above the minima, and the result follows.  $\square$

**Proposition 3.12.** *If  $T_s$  is graded with length 1, the Hasse diagram is a line.*

*Proof.* Suppose  $s$  has  $n$  1s and  $n$  0s. Rotate  $s$  so that the first bit is a 1 and the last is a 0. This either has no impact on  $T_s$ , or inverts it. From Proposition 3.10,  $s$  must consist of  $k$  1s, then  $l$  0s,  $n - k$  1s, then  $n - l$  0s, for some positive integer  $k$  and  $l$ .

Suppose  $l$  is greater than  $k$ . Consider the  $(2k + 1)$ th bit. This is a 0 which must pair to a 1 such that the substring between them must be balanced, since bits in that substring cannot pair outside of it. Then

it cannot pair to the first  $k$  1s, and in fact it must pair to the  $2l$ th bit. Likewise, the  $(2k + i)$ th bit must pair to the  $(2l - i + 1)$ th bit, for  $2k < 2k + i \leq l$ . Since these bits always pair to each other, the corresponding blocks are in each element of  $T_s$ . Removing these produces a poset isomorphic to the original one. Thus we may assume without loss of generality that  $l$  is smaller than or equal to  $k$ . Likewise, we can rotate the first  $k$  bits, and by the same argument, we can assume that  $n - k$  is smaller than or equal to  $l$ , and so on. Then we have that  $k \geq l \geq n - k \geq n - l \geq k$ . Then they are all equal, and we may assume without loss of generality that  $s$  consists of  $k$  1s,  $k$  0s,  $k$  1s, then  $k$  0s.

If all of the bits in the first series of 1s pairs to the first series of 0s, then the  $i$ th bit for  $1 \leq i \leq k$  pairs to the  $(2k - i + 1)$ th bit, as that is the only bit for which the substring between the paired bits is balanced. Likewise for the  $k$  bits in the second series of 1s. Call the element in  $T_s$  that corresponds to this pairing,  $a$ . Note that if the first bit pairs to the  $2k$ th bit, then the first  $2k$  bits pair within themselves, so it has to give the above pairing. Then any other element of  $T_s$  must have an underlying pairing which pairs the first bit to some other bit, and the only possible choice is the  $4k$ th bit.

Now I will prove the proposition by induction on  $k$ . The base case is trivial, as it has only two partitions, one greater than the other. Now suppose that it is true for positive integers less than  $k$ , for  $k$  greater than 1. For each element of  $T_s$  not  $a$ , the first bit pairs to the last. Since the  $2k$ th bit can only pair to the first or  $(2k + 1)$ th bit, it must pair to the  $(2k + 1)$ th bit. Now rotate the first bit. This inverts the poset, but we will see that it does not matter. Now all such elements have the same two singletons, and the poset formed by those is isomorphic to the poset formed from the string with those bits removed, which then consists of  $k - 1$  1s,  $k - 1$  0s,  $k - 1$  1s, then  $k - 1$  0s. By our induction assumption, the Hasse diagram is a line. Then the subset of elements in  $T_s$  not  $a$  has a Hasse diagram which is an inverted line, which is still a line.

Consider the relation of  $a$  to the other elements. It conforms to the structure from Theorem 3.9, so it is a minimum. An element greater than it must then have the same structure as  $a$ , except for two blocks which are joined together. Let  $b$  be such an element. As noted above,  $b$  must have an underlying pairing which pairs the first bit with the last, and the  $2k$ th bit with the  $(2k + 1)$ th bit. If the second bit pairs to the second last bit, by a similar argument to the above, the  $(2k - 1)$ th

bit pairs with the  $(2k + 2)$ th bit. Then there is a 2-block in  $b$  consisting of the  $k$ th and  $(k + 1)$ th elements. But since the first and  $k$ th elements forms a 2-block in  $a$ ,  $b$  cannot be greater than  $a$ . Therefore the second bit cannot pair to the second last bit. The second bit must then pair to the  $(2k - 1)$ th bit. Each  $i$ th bit for  $1 < i \leq k$  must thus pair to the  $(2k - i + 1)$ th bit, and likewise, each  $(2k + i)$ th bit for  $1 < i \leq k$  must pair to the  $(4k - i + 1)$ th bit. Therefore, this is the only possible pairing for  $b$ . Indeed,  $b$  is greater than  $a$ , and so there is exactly one element greater than  $a$ . From Lemma 3.11,  $b$  is greater than at most one other element. The subposet of elements of  $T_s$  not  $a$  has a Hasse diagram which is a line. Since it is connected,  $b$  is greater than some element. From the above, it is greater than only one element, so  $b$  is at the end of the line. As  $a$  is less than  $b$  and has no relation to the other elements,  $T_s$  has a Hasse diagram which is a line. By induction, this is true for all  $k$ .  $\square$

*Example 3.13.* Referring to Example 3.7, with  $\mathbf{s} = 1011010010$  we have  $\delta(\mathbf{s}) = \diamond 1 \diamond 0 \diamond$ . The only partition in  $NC_{1,2}(\diamond 1 \diamond 0 \diamond)$  is the partition  $\pi_0$  in Figure 7, confirming Theorem 3.9 in this case. On the other hand, with  $\mathbf{t} = 111000111000$  so that  $\delta(\mathbf{t}) = 1 \diamond 0 1 \diamond 0$ , there are two partitions in  $NC_{1,2}(1 \diamond 0 1 \diamond 0)$ , pictured below in Figure 18. Hence,  $T(111000111000)$  has two distinct minimal elements.

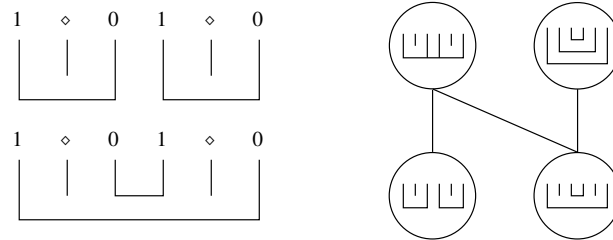


FIGURE 18. There are two partitions in  $NC_{1,2}(1 \diamond 0 1 \diamond 0)$ , pictured above. Also shown is the poset  $T(111000111000)$ , where they appear as the minima.

#### 4. CONNECTEDNESS

A poset  $P$  is *connected* if for all  $a, b \in P$  there exists a sequence  $a = x_0, x_1, \dots, x_m = b$  in  $P$  such that for all  $0 \leq i \leq m - 1$  the elements  $x_i$  and  $x_{i+1}$  are comparable. Our goal in this section is to prove the following.

**Theorem 4.1.** *The poset  $T_s$  is connected.*

The proof will proceed by induction on the size of  $s$  with the initial step being obvious. As any element lies above some minima it is sufficient to show that any two minima are connected by a sequence of comparable elements.

**Definition 4.2.** Two minima are immediately related if they differ in only two blocks. Two minima  $m_1$  and  $m_2$  are related if there is a sequence of minima starting with  $m_1$  and ending with  $m_2$  such that each two consecutive minima are immediately related.

By the proof of Theorem 3.9, two distinct minima  $m_1$  and  $m_2$  have the same 1-blocks. If a 2-block  $a_1$  in  $m_1$  is not in  $m_2$ , then the first element of  $a_1$  is paired in  $m_2$  to another element  $x$ , and similarly the second element is paired to some other element  $y$ . If they are immediately related, then they cannot differ in any more blocks, so  $x$  is paired to  $y$  in  $m_1$ . Hence, immediately related minima have blocks which are the same except for the configuration shown in Figure 19.

**Lemma 4.3.** Two immediately related minima have a join.

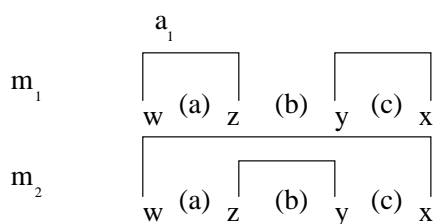


FIGURE 19.

*Proof.* In the lattice of partitions of  $n$  elements the join of two immediately related minima  $m_1$  and  $m_2$  is as shown in Figure 20, with  $w$ ,  $x$ ,  $y$ , and  $z$  in the same block as depicted, and where  $a$ ,  $b$ , and  $c$  represent some collections of other blocks in the partition.

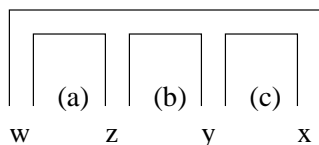


FIGURE 20.

From the structure of  $m_1$ , we know that elements in  $a$  and  $c$  pair only to themselves, as  $m_1$  is non crossing. Similarly, from  $m_2$ , we

see that the same is true for  $b$ . So the above join is also the join in  $NC(n)$ . For each of the four elements  $w, x, y,$  and  $z$ , since they are part of a 2-block in a minima, the two bits underlying each of them are either both 1s or both 0s. Without loss of generality, suppose that  $w$  is 1 in the constructed substring. Because  $z$  is paired to it in  $m_1$ ,  $z$  is 0. Similarly,  $y$  is 1 and  $x$  is 0. Thus there is a pairing of the original bitstring which maps to this partition, shown in Figure 21. Therefore, it is a valid partition and is the join of the two minima in  $T_s$ . Hence two immediately related minima are connected, and so related minima are connected.  $\square$

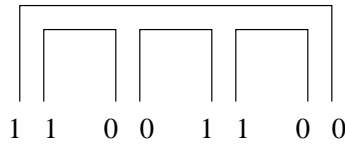


FIGURE 21.

**Lemma 4.4.** *Any two distinct minima in  $T_s$  are related.*

*Proof.* Let  $\delta(s)$  with the diamonds omitted be  $s_1$ . Then  $s_1$  has at most  $n/2$  1s, so by induction assumption  $T_{s_1}$  is connected.

Every minima in  $T_s$  can be uniquely represented as an element in  $T_{s_1}$ , as they correspond to the pairings of  $s_1$ , which map bijectively to the elements of  $T_{s_1}$ . For each minima  $m$  in  $T_s$ , let  $f(m)$  be the element of  $T_{s_1}$  that corresponds to it.

Let  $m_1$  and  $m_2$  be two minima such that  $f(m_1)$  covers  $f(m_2)$  in  $T_{s_1}$ . Then some block in  $f(m_1)$  is broken up into two blocks in  $f(m_2)$ , either one nested in the other, or not. The case where it is not nested is shown in Figure 22.

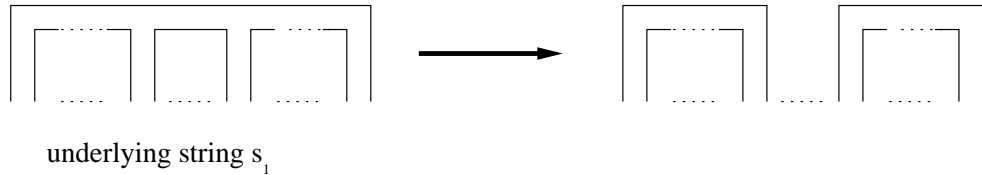


FIGURE 22.

Thus  $m_1$  and  $m_2$  are immediately related, as they only differ in two 2-blocks as shown in in Figure 23. The other case is similar.

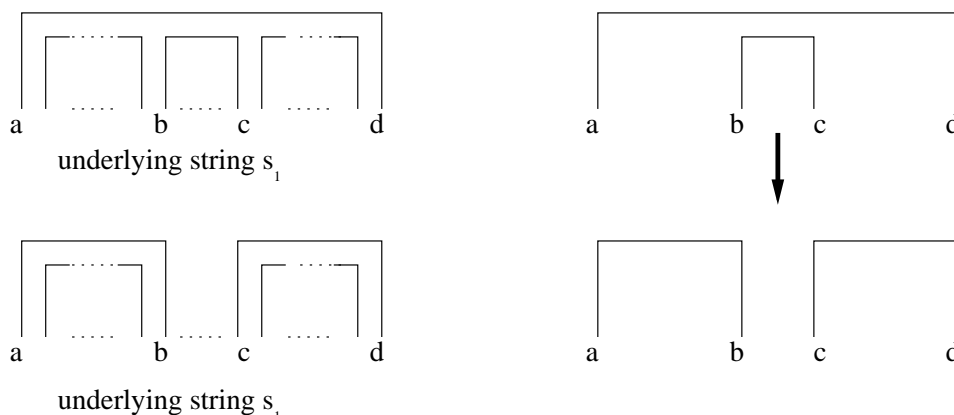


FIGURE 23.

Since  $T_{s_1}$  is connected, for any minima  $m$  and  $m'$  of  $T_s$  we can find a sequence of elements in  $T_{s_1}$  starting with  $f(m)$  and ending with  $f(m')$  such that each one covers or is covered by the next. Therefore  $m$  and  $m'$  are related.  $\square$

I will be using the concept of a "hill diagram" here, as a way of representing a bitstring in a way that makes it easy to see which 1s can pair to which 0s. Suppose we have a balanced bitstring  $s$  of length  $2n$ . Consider an integer grid with the following points.  $(0,0)$  is the first point, and the  $(i+1)$ th point for  $1 \leq i \leq 2n$  is  $(i, f(i))$  where  $f(i)$  is the number of 1s minus the number of 0s in the first  $i$  bits of  $s$ .

**Definition 4.5.** *The hill diagram of  $s$  is the diagram formed by joining the consecutive points above. The bit of  $s$  that corresponds to the  $i$ th edge is the  $i$ th bit.*

Notice that the final point must be  $(2n, 0)$ , as the bitstring is balanced.

**Proposition 4.6.** *A 1 and 0 of a bitstring  $s$  is paired in some noncrossing pairing of  $s$  if and only if their corresponding edges are between  $y = k$  and  $y = k + 1$  for some integer  $k$ .*

*Proof.* If there is a noncrossing pairing of  $s$  such that the  $i$ th bit is paired to the  $j$ th bit for  $i < j$ , then the bits in between them must form a balanced substring, as they must pair to themselves in that noncrossing pairing. If the  $i$ th bit is a 0, then the  $j$ th bit is a 1. Suppose the  $(i+1)$ th point is  $(i, k)$ . Since the next  $j-i-1$  bits form a balanced substring, the  $j$ th point is  $(j-1, k)$ . Then the  $i$ th and  $j$ th edges are between  $y = k$  and  $y = k + 1$ . Likewise if the  $i$ th bit is a 1,

then the  $j$ th bit is a 0.

Now suppose that the  $i$ th and  $j$ th edges are between  $y = k$  and  $y = k + 1$  for some integer  $k$ . Suppose the  $i$ th bit is 0 and  $j$ th bit is 1. Then the  $(i + 1)$ th point is  $(i, k)$  and the  $j$ th point is  $(j - 1, k)$ . Then the bits between them forms a balanced substring, so there is a noncrossing pairing of those bits. The rest of the bits are also balanced, so there is a noncrossing pairing of those bits. Pair the  $i$ th and  $j$ th bit, and pick a noncrossing pairing of the bits between them, and a noncrossing pairing of the rest of the bits. Since each part pairs within itself, no crossings are introduced, and this is a noncrossing pairing of  $s$ . Likewise if the  $i$ th bit is 1 and the  $j$ th bit is 0. Then the result holds.  $\square$

Note that we can view edges that are between some  $y = k$  and  $y = k + 1$  as being "on the same level".

**Theorem 4.7.** *Up to switching 1s with 0s, the bitstring  $s$  associated with such a poset of partitions  $T_s$  is unique.*

*Proof.* First note that if we flip the 1s and 0s of a bitstring, the pairings are exactly the same, so the poset is the same. Thus the poset associated with a bitstring starting with a 0 is exactly the same as that for the corresponding bitstring that starts with a 1. Suppose that for bitstrings  $s$  and  $s'$ ,  $T_s = T_{s'}$ . Without loss of generality, let the first element of  $s'$  be 1. In the hill diagrams mentioned earlier, we know that if there is a 1 and a 0 on the same level, there is a pairing which pairs them. Each partition is mapped from some pairing of the bitstring, so  $T_s$  corresponds to a set of pairings. Then  $s'$  and  $s$  each have this set of pairings. If we have a 1 on some level of the hill diagram,  $s'$  must have a 0 on the same level at a position  $x$  if there is some pairing of  $s'$  which pairs that 1 to position  $x$ . Conversely, if there is no such pairing at that position, then there cannot be a 0 of that level in that position. We can do a similar thing with 0s. By starting with the first element, we can determine all the 0s on the first level. The first element must have at least one possible pairing, so there is at least one 0 on the first level. Picking that, we can determine all the 1s on the first level. Then we look at the first undetermined element as shown in Figure 24.

If the element before that is a 1 (up) then it cannot be a 0 (down) and vice versa, or else it would be on the same level on hence be determined. So we can determine that element, and then all the elements on that level. Since there are a finite number of elements,

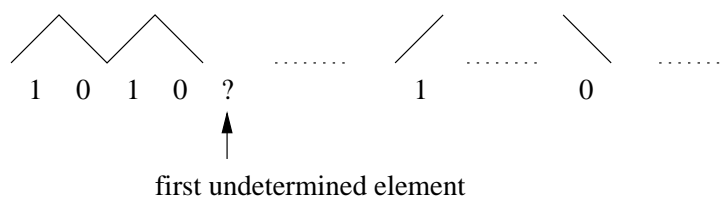


FIGURE 24.

this process must end eventually, giving us a complete, fully determined bitstring. Thus there is only one bitstring starting with 1 that is mapped to any such poset of partitions. Any bitstring starting with 0 that yields the same poset, when 1s and 0s are flipped, must therefore be that bitstring. So the bitstring associated with such a poset is uniquely determined.  $\square$

Note that there are many sets of partitions that are not mapped onto by any bitstring, and that there are many distinct sets of partitions which are isomorphic as posets under refinement. For example, as noted before, a rotation by two bits will produce a poset which is isomorphic, even though the set of partitions is in general different.

## 5. CONSTRUCTIBILITY

I will now show that  $T_s$  is constructible. Since  $T_s$  is a finite poset, the posets in the results of this section will be taken to be finite posets.

**Definition 5.1.** *Define the class of constructible simplicial complexes recursively as follows:*

- 1) *Every simplex including  $\emptyset$  and  $\{\emptyset\}$  is constructible.*
- 2) *If  $A$  and  $B$  are constructible complexes of dimension  $d$  and the intersection of  $A$  and  $B$  is a constructible complex of dimension  $d - 1$ , then the union of  $A$  and  $B$  is constructible.*

Constructible complexes were introduced by Hochster [8].

**Definition 5.2.** *The order complex of a poset  $P$  is the simplicial complex whose vertices are the elements of  $P$ , and whose faces are the chains of  $P$ . [7]*

**Definition 5.3.** *A poset is constructible if its order complex is constructible.*



I will need the definition of recursive atom orderings [9], which I will abbreviate as "RA-ordering".

**Definition 5.4.** *A bounded poset  $P$  admits an RA-ordering if its length is 1, or its length is greater than 1 and there is an ordering  $a_1, a_2, \dots, a_n$  of the atoms of  $P$  such that:*

- 1) *For all  $j = 1, 2, \dots, n$  the interval  $[a_j, 1]$  admits an RA-ordering in which the atoms of  $[a_j, 1]$  that belong to  $[a_i, 1]$  for some  $i < j$  come first.*
- 2) *For all  $i < j$ , if  $a_i, a_j < y$  then there is a  $k < j$  and an atom  $z$  of  $[a_j, 1]$  such that  $a_k < z \leq y$ .*

**Definition 5.5.** *Let  $A'$  be  $A \cup \{0\}$ .*

**Theorem 5.6.** *Let  $A$  and  $B$  be two join-semilattices such that  $A'$  and  $B'$  admit RA-orderings. Then  $(A \times B)'$  admits an RA-ordering.*

Before proving this, I will first need the following lemma.

**Lemma 5.7.** *Let  $A$  and  $B$  be as in the above theorem. For any  $x$  not 1 in  $A$ ,  $y$  not 1 in  $B$ , if  $a_1, a_2, \dots, a_n, a'_1, a'_2, \dots, a'_n$  is an RA-ordering of  $[x, 1]$ , and  $b_1, b_2, \dots, b_m, b'_1, b'_2, \dots, b'_m$  is an RA-ordering of  $[y, 1]$ , then there is an RA-ordering of  $[(x, y), 1]$  such that the atoms  $(a_i, y)$  and  $(x, b_j)$  come first.*

*Proof.* I will show this by induction on the size of  $[(x, y), 1]$ . The base case is where  $x$  and  $y$  are coatoms of  $A$  and  $B$  respectively, and it is true.

Order the atoms of  $[(x, y), 1]$  as follows.  $(a_1, y), (a_2, y), \dots, (a_n, y), (x, b_1), (x, b_2), \dots, (x, b_m), (a'_1, y), (a'_2, y), \dots, (a'_n, y), (x, b'_1), (x, b'_2), \dots, (x, b'_m)$ . I will show that it satisfies the definition of RA-orderings part by part. For each  $(a_{i'}, y)$ , the atoms of  $[(a_{i'}, y), 1]$  that are above atoms of  $[(x, y), 1]$  that precede  $(a_{i'}, y)$  are exactly the elements  $(z_i, y)$  where the  $z_i$  are the atoms of  $[a_{i'}, 1]$  that are above some  $a_{i''}$ ,  $i'' < i'$ . (If  $x$  is a coatom, then  $n \leq 1$ , so either this part does not exist, or consists only of  $(1, y)$ .  $[(1, y), 1]$  is isomorphic to  $[y, 1]$ , and so admits an RA-ordering.) Since  $[(a_{i'}, y), 1]$  does not include  $(x, y)$ , it has a smaller size than  $[(x, y), 1]$ . From the induction assumption, it has an RA-ordering beginning with the elements  $(z_i, y)$ . Also, if  $(a_{i''}, y)$  and  $(a_{i'}, y)$  for some  $i'' < i'$  are below some element  $(e_1, e_2)$ , then  $a_{i'}$  and  $a_{i''}$  are below  $e_1$ . Then  $a_{i'''}$  for some  $i''' < i'$  and an atom  $e'_1$  of  $[a_{i'}, 1]$  exist such that  $a_{i'''} < e'_1 \leq e_1$ , so  $(a_{i'''}, y)$  for some  $i''' < i'$  and an atom  $(e'_1, y)$  of  $[(a_{i'}, y), 1]$  exist such that  $(a_{i'''}, y) < (e'_1, y) \leq (e_1, e_2)$ .

Now consider each  $(x, b_{j'})$ . Suppose there is an element  $e$  which is above  $(x, b_{j'})$  and some atom  $z$  preceding it. If  $z$  is some  $(x, b_j)$  then we repeat the above argument. If  $z$  is some  $(a_i, y)$ , then let  $e = (e_1, e_2)$ .  $e_1 \geq x$  and  $a_i$ , and  $e_2 \geq y$  and  $b_{j'}$ , so  $(a_i, y) < (a_i, b_{j'}) \leq e$ , and  $(a_i, b_{j'})$  is an atom of  $(x, b_{j'})$ . Thus the second property holds. Since  $(a_i, b_{j'})$  covers  $(a_i, y)$  and  $(x, b_{j'})$ , it is in fact their join. Then the atoms of  $[(x, b_{j'}), 1]$  are all the  $(a_i, b_{j'})$  as well as the elements  $(x, z_i)$  where the  $z_i$  are the atoms of  $[b_{j'}, 1]$  that are above some  $b_{j''}$ ,  $j'' < j'$ . The latter type does not exist if  $b_{j'} = 1$ , of course. In that case,  $[(x, 1), 1]$  is isomorphic to  $[x, 1]$ , so an ordering of the atoms exist with the  $(a_i, 1)$  coming first. Otherwise, we apply the induction assumption to  $[(x, b_{j'}), 1]$ , as the  $a_i$  can come first in an RA-ordering of  $[x, 1]$ , and the  $z_i$  can come first in an RA-ordering of  $[b_{j'}, 1]$ . Then the first property is also satisfied.

Now consider each  $(a'_i, y)$ . The same argument used above applies to this case as well, since there are the same two types of atoms preceding it, those of the form  $(z, y)$  where  $z$  precedes  $a'_i$  in the RA-ordering of  $[x, 1]$ , and  $(x, z)$ , where the "z"s are a set of atoms of  $[y, 1]$  that can come first in its RA-ordering. The same is true of the atoms of the form  $(x, b'_{j'})$ . Then it is true for  $[(x, y), 1]$  as well, so it is true for any pair of elements from  $A$  and  $B$ .  $\square$

Notice that if you set  $x = 0$  and  $y = 0$ , this implies that products of bounded join-semilattices admitting RA-orderings have an RA-ordering that starts with atoms of the form  $(a, 0)$  and  $(0, b)$ , where the atoms  $a$  of the first bounded join-semilattice come first in some RA-ordering, and likewise for  $b$ .

Now we are ready to prove the theorem.

*Proof.* Let an RA-ordering of  $A'$  be  $\alpha_1, \alpha_2, \dots, \alpha_k$ , and an RA-ordering of  $B'$  be  $\beta_1, \beta_2, \dots, \beta_l$ . Express the elements of  $(A \times B)$  as pairs  $(c, d)$  where  $c$  in  $A$ ,  $d$  in  $B$ . Then the atoms of  $(A \times B)'$  are exactly the elements  $(\alpha_i, \beta_j)$  for some  $1 \leq i \leq k, 1 \leq j \leq l$ . Now order these in the following way. The first atom is  $(\alpha_1, \beta_1)$ , and for each  $(\alpha_i, \beta_j)$ , the next atom is  $(\alpha_{i-1}, \beta_{j+1})$  if it exists, and if it does not, the next atom is  $(\alpha_{i'}, \beta_{j'})$  where  $i' + j' = i + j + 1$  such that  $j'$  is the smallest possible number such that  $(\alpha_{i'}, \beta_{j'})$  exists. This terminates at  $(\alpha_k, \beta_l)$ , and the order is  $(\alpha_1, \beta_1), (\alpha_2, \beta_1), (\alpha_1, \beta_2), (\alpha_3, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_k, \beta_{l-1}), (\alpha_{k-1}, \beta_l), (\alpha_k, \beta_l)$ .

I will show that this is indeed an RA-ordering. Any atom  $z'$  of  $(A \times B)'$  is some  $(\alpha_{i'}, \beta_{j'})$  for  $i'$  and  $j'$  satisfying  $1 \leq i' \leq k, 1 \leq j' \leq l$ . Then any atom  $z = (\alpha_i, \beta_j)$  preceding  $z'$  satisfies  $i + j \leq i' + j'$ . If both atoms are below some element  $(e_1, e_2)$ , then  $e_1 > \alpha_{i'}, \alpha_i, e_2 > \beta_{j'}, \beta_j$ . If  $j < j'$ ,  $(e_1, e_2) > (\alpha_{i'}, \beta_j)$ . From the RA-ordering of  $B'$ , there exists  $j'' < j'$  and an atom  $z_0$  of  $[\beta_{j'}, 1]$  such that  $\beta_{j''} < z_0 \leq e_2$ . Then  $(\alpha_{i'}, \beta_{j''})$  is an atom preceding  $z'$  in the ordering, and  $(\alpha_{i'}, z_0)$  is an atom of  $[(\alpha_{i'}, \beta_{j''}), 1]$ , such that  $(\alpha_{i'}, \beta_{j''}) < (\alpha_{i'}, z_0) \leq (e_1, e_2)$ . On the other hand, if  $j \geq j'$ , we have that  $i < i'$ . By following the same argument, we get the same result, so the second property of Definition 5.4 holds.

Suppose an atom of  $[(\alpha_{i'}, \beta_{j'}), 1]$  belongs to some  $[(\alpha_i, \beta_j), 1]$  for a  $(\alpha_i, \beta_j)$  that precedes  $(\alpha_{i'}, \beta_{j'})$  in the ordering. From the above argument, it is equal to or above some atom (thus is actually equal to)  $(\alpha_{i'}, z_0)$  or  $(z_0, \beta_{j'})$ , which in turn is above some  $(\alpha_{i'}, \beta_{j''})$  or  $(\alpha_{i''}, \beta_{j'})$  with  $j'' < j'$  and  $i'' < i'$  respectively. On the other hand, for each atom  $z_0$  of  $[\beta_{j'}, 1]$  that is above some  $\beta_{j''}$  for  $j'' < j'$ ,  $(\alpha_{i'}, z_0)$  is an atom of  $[(\alpha_{i'}, \beta_{j'}), 1]$  that is above  $(\alpha_{i'}, \beta_{j''})$ , which comes before  $z'$  in the ordering. Likewise for atoms of  $[\alpha_{i'}, 1]$  which are above some  $\alpha_{i''}$  for  $i'' < i'$ . Then the atoms of  $[(\alpha_{i'}, \beta_{j'}), 1]$  which are above some  $(\alpha_i, \beta_j)$  that precedes  $(\alpha_{i'}, \beta_{j'})$  in the ordering are exactly those  $(\alpha_{i'}, z_0)$  for which  $z_0$  is an atom of  $[\beta_{j'}, 1]$  that is above some  $\beta_{j''}$  for  $j'' < j'$ , and  $(z_1, \beta_{j'})$  for which  $z_1$  is an atom of  $[\alpha_{i'}, 1]$  that is above some  $\alpha_{i''}$  for  $i'' < i'$ .

From the definition of RA-orderings, the atoms of  $[\alpha_{i'}, 1]$  can be ordered as  $a_1, a_2, \dots, a_n, a'_1, a'_2, \dots, a'_n$ , such that the atoms that are above some  $\alpha_{i''}$  for  $i'' < i'$  are exactly the atoms  $a_1, a_2, \dots, a_n$ . Likewise, the atoms of  $[\beta_{j'}, 1]$  can be ordered as  $b_1, b_2, \dots, b_m, b'_1, b'_2, \dots, b'_m$ , such that the atoms that are above some  $\beta_{j''}$  for  $j'' < j'$  are exactly the atoms  $b_1, b_2, \dots, b_m$ . Then the atoms of  $[z', 1]$  which are above some  $z$  that precedes  $z'$  in the ordering are the atoms  $(\alpha_{i'}, b_j)$   $(\alpha_i, \beta_{j'})$ , for positive integers  $i \leq n, j \leq m$ . Applying Lemma 5.7, these can come first in an RA-ordering of  $[z', 1]$ . Thus the first property is also true, and therefore  $(A \times B)'$  admits an RA-ordering.  $\square$

We can also have a variation of Lemma 5.7.

Let  $A$  and  $B$  be two posets such that  $A$  and  $B$  with a top elements added are join semi-lattices. Let  $A''$  be  $A$  with a bottom and top element added, and likewise for  $B''$ . Suppose  $A''$  and  $B''$  admit RA-orderings.

**Lemma 5.8.** *For any  $x$  in  $A$ ,  $y$  in  $B$ , if  $a_1, a_2, \dots, a_n, a'_1, a'_2, \dots, a'_n$  is an RA-ordering of  $[x, 1]$ , and  $b_1, b_2, \dots, b_m, b'_1, b'_2, \dots, b'_m$  is an RA-ordering of  $[y, 1]$ , then there is an RA-ordering of  $[(x, y), 1]$  such that the atoms  $(a_i, y)$  and  $(x, b_j)$  come first.*

Note that  $(1, y)$  does not exist in the product of  $A$  and  $B$  as the  $1$  does not exist in  $A$  itself. Likewise for  $(x, 1)$ . The proof then follows the same proof in Lemma 5.7. The corresponding version of Theorem 5.6 also follows in the same way.

A few more properties should be noted here.

**Proposition 5.9.** *For any two elements in  $T_s$  with at least one element greater than both, the join is the join of the two elements in  $NC(n)$ . Likewise, for any two elements in  $T_s$  with at least one element less than both, the meet is the meet of the two elements in  $NC(n)$ .*

*Proof.* Let  $x$  and  $y$  be two elements of  $T_s$  such that both are less than or equal to some  $z$  in  $T_s$ . Let the join of  $x$  and  $y$  be  $z'$ . Then  $z' \leq z$ . Since  $T_s$  is convex in  $NC(n)$ ,  $z'$  is in  $T_s$ . Any element greater than or equal to both  $x$  and  $y$  in  $T_s$  is greater than or equal to  $z'$ , which itself is greater than or equal to both  $x$  and  $y$ . Thus  $z'$  is also the join of both  $x$  and  $y$  in  $T_s$ . Likewise for meets.  $\square$

This means that any  $T_s$  with a top element is a join semilattice. In fact, for any  $T_s$ , adding a top element if there are more than one maximum, and adding a bottom element if there are more than one minimum, transforms it into a lattice. This is because from the above proposition, if there are elements above/below some  $x$  and  $y$ , the join/meet exists, and if not, then the top/bottom element is the join/meet.

Let the poset of partitions of  $NC(2n)$  that have even block size be  $NC^2(2n)$ . Then the atoms of  $NC^2(2n)'$  are non crossing pairings of  $2n$  elements. For such an atom  $z$ , the atoms of  $[z, 1]$  are obtained by joining two such pairs into a block. If those two pairs are  $(a, d)$  and  $(b, c)$  where  $a < b < c < d$ , then call such an atom of  $[z, 1]$  a nest-atom, which I will abbreviate as "N-atom".

**Theorem 5.10.**  *$NC^2(2n)'$  admits an RA-ordering.*

Before proving this, I will need the following lemma.

**Lemma 5.11.** *For  $n > 1$ ,  $NC(n)$  has an RA-ordering where the atoms are ordered according to lexicographic order of the atom's 2-block ordered as the smaller element first then the larger.*

*Proof.* I will prove this by induction on  $n$ . The base case is trivial. Suppose it is true for any integer from 2 up to  $n - 1$ . Then I will show that it holds for  $n$  as well.

Take any atom  $z$ . Its 2-block is some  $(i, j)$  for  $i < j$ . Pick an atom  $z'$  that precedes it. then its 2-block is some  $(i', j')$  for  $i' < j'$ . If  $i' = i$ , then  $j' < j$ , and its join is an atom of  $[z, 1]$ , satisfying the second property. This join then has a 3-block with elements  $i < j' < j$ , with the rest singletons. If  $i' < i$  and  $i < j' < j$ , then the join of  $z$  and  $z'$  is above some join of the above type, so again, the second property is satisfied. Otherwise,  $i' < i$  and  $j' \leq i$  or  $j' \geq j$ . In this case, the join of  $z$  and  $z'$  is an atom of  $[z, 1]$  too, satisfying the second property. This either forms a 3-block  $i' < i < j$ , or two 2-blocks,  $(i, j)$  and  $(i', j')$ . Each of this is the join of  $z$  and  $z'$  for some  $z'$  preceding  $z$ .

Now I show that  $[z, 1]$  is isomorphic to a product of  $NC(j - i)$  and  $NC(n - j + i)$ . Map each element  $x$  in  $[z, 1]$  to a pair  $(a(x), b(x))$  in  $(NC(j - i), NC(n - j + i))$  as follows.  $a(x)$  is obtained by deleting elements 1 to  $i - 1$ , and  $j$  to  $n$ .  $b(x)$  is obtained by deleting elements  $i$  to  $j - 1$ . Since  $i$  and  $j$  are always in the same block, no other block can contain elements inside and outside this pair. Any element which maps to  $(a(x), b(x))$  must therefore have exactly the blocks in  $a(x)$  and  $b(x)$ , except that the block containing  $i$  in  $a(x)$  must be joined to the block containing  $j$  in  $b(x)$ . Then this map is injective. By the same argument, any pair  $(c, d)$  can be mapped to from some element in  $NC(n)$ . Since  $i$  and  $j$  are in the same block, it is in  $[z, 1]$ . Thus the map is bijective.

The atoms of  $[z, 1]$  that are greater than some atom  $z'$  preceding  $z$  in the ordering are of the form  $(e, 0)$  and  $(0, f)$ . The  $e$  are atoms of  $NC(j - i)$  where the 2-block is a pairing of the first element,  $i$ , with any other element. This is a set that occurs first in the lexicographic order of the atoms mentioned above, and by induction assumption, there is an RA-ordering beginning with it. The  $f$  are atoms of  $NC(n - j + i)$  whose 2-block has a first element that is before  $i$ . This is also a set that comes first in the lexicographic order of the atoms mentioned above, so by induction assumption, there is an RA-ordering beginning with it. From the proof of lemma 5.7, there is an RA-ordering of  $[z, 1]$

that puts the set of atoms of the form  $(e, 0)$  and  $(0, f)$  first. Thus this satisfies the first property for RA-orderings, and hence is true for  $n$  as well. Then the result holds.  $\square$

Now we prove the theorem.

*Proof.* I will show that  $NC^2(2n)'$  admits an RA-ordering, and for each atom  $z$  of  $NC^2(2n)'$  for  $n > 1$ , there is an RA-ordering of the atoms of  $[z, 1]$  such that the N-atoms come first, by induction on  $n$ . The base cases are  $n = 1$  and  $n = 2$ , and are trivially true. Now suppose that the above is true for integers less than  $n$  and greater than one, and I will show that it holds for  $n$  as well.

I will use the atom ordering used in Wachs [9] for partitions with even block size, and show that this in fact works for  $NC^2(2n)'$  too. For each atom of  $NC^2(2n)'$ , associate the following word to it. Order each pair in increasing order of its two elements, and list the pairs in lexicographic order. Order the atoms in lexicographic order of the words. Let  $z$  be some atom of  $NC^2(2n)'$ , and  $z'$  be an atom that precedes it in the ordering. Suppose their words first differ on the  $k$ th pair. Then the words up to that pair agree. For any word, if  $i$  is the smallest positive integer yet to be listed up to the  $j$ th pair, then the pair containing that must be the next to be listed, so the  $(j + 1)$ th pair consists of  $i$  and some higher element. Thus the  $k$ th pair of both  $z$  and  $z'$  must contain some common element  $a$  and some different higher element. Suppose the  $k$ th pair of  $z$  is  $(a, b)$  and that of  $z'$  is  $(a, b')$ . Since  $z'$  precedes  $z$ ,  $b' < b$ . Note that  $NC^2(2n)$  is  $T_{(1100)^n}$ , and has a top element, so  $NC^2(2n)'$  is a lattice. Then from Proposition 5.9, the join of  $z$  and  $z'$  exists and is their join in  $NC(2n)$ . This join thus has one block  $A$  that contains at least  $a, b$ , and  $b'$ .

For any pair with an element to the left of  $a$ , it appears in both  $z$  and  $z'$ , so the elements between the two elements of the pair must pair to themselves in both  $z$  and  $z'$ , and likewise for the other elements. Thus an element to the left of  $a$  cannot be in  $A$ . Then the leftmost element of  $A$  is  $a$ . Notice that in either  $z$  or  $z'$ , the elements between any two consecutive elements of  $A$  must pair to themselves, or else there is a block that crosses  $A$ . Let the second leftmost element of  $A$  be  $c$ .  $b$  is not the second leftmost element of  $A$ , and since it pairs to  $a$  in  $z$ ,  $c$  must pair to another element of  $A$  between  $a$  and  $b$  in  $z$ . Let this be  $d$ . This is to the right of  $c$  and to the left of  $b$ , as  $c$  is the second leftmost element of  $A$ . Let  $z''$  be the partition whose pairs are the same as that of  $z$ , except that instead of the pairs  $(a, b)$

and  $(c, d)$ , it pairs  $a$  with  $c$  and  $d$  with  $b$ . Since the elements between any two consecutive elements of  $A$  must pair to themselves, this is non crossing, and so  $z''$  is an atom of  $NC^2(2n)'$ . Since  $c$  precedes  $b$ ,  $z''$  precedes  $z$  in the ordering of atoms.

The join of  $z''$  with  $z$  consists of the other pairings of  $z$  together with the block with elements  $a, c, d, b$ . Each 2-block is in a block of  $z$ , so it is in some block in the join of  $z$  and  $z'$ . The block with elements  $a, c, d, b$  is also in  $A$ , which is in the join of  $z$  and  $z'$ . Then the join of  $z$  and  $z''$  is less than or equal to any element that is greater than or equal to both  $z$  and  $z'$ . This is an atom of  $[z, 1]$ , and  $z''$  precedes  $z$  in the ordering, so this satisfies property two of the RA-ordering. Notice that this is an N-atom of  $[z, 1]$ . Then any atom of  $[z, 1]$  belonging to some  $[z', 1]$  for  $z'$  preceding  $z$  is an N-atom.

Now I will show that all N-atoms of  $[z, 1]$  belong to some  $[z'', 1]$  for  $z''$  preceding  $z$ . Pick any N-atom of  $z$ . Then the blocks of that N-atom are the pairs of  $z$  except for one block containing elements  $a < b < c < d$  where  $(a, d)$  and  $(b, c)$  are pairs of  $z$ . Then let  $z''$  be the atom whose pairs are the same as that of  $z$ , except that instead of the pairs  $(a, d)$  and  $(b, c)$ , it pairs  $(a, b)$  and  $(c, d)$ . Then that N-atom belongs to  $z''$ . Their words first differ in the pairs  $(a, b)$  and  $(a, d)$ , so  $z''$  precedes  $z$ . Thus the atoms of  $[z, 1]$  that are above some atom of  $NC^2(2n)'$  that precedes it in the ordering are exactly the N-atoms. By our induction assumption, there is an RA-ordering of  $[z, 1]$  where the N-atoms come first. Then the first property holds, and this ordering of the atoms of  $NC^2(2n)'$  is an RA-ordering.

Now I will show that there is an RA-ordering of the atoms of  $[z, 1]$  such that the N-atoms come first. Consider  $[z, 1]$ . We have two cases. First, we can find some element  $i$  that pairs to some  $j$  greater than  $i$  in  $z$ , such that their difference is greater than one, and  $(i, j)$  is not  $(1, 2n)$ . Second, this is not true, in which case every element from 2 up to  $2n - 1$  is paired with an adjacent element. If 2 is paired with 1, then 3 is paired with 4, and so on, up to  $2n - 1$  and  $2n$ . Then there are no N-atoms and the assertion that there is an RA-ordering of the atoms of  $[z, 1]$  such that the N-atoms come first, is trivially true. If 2 is paired to 3, then 4 is paired to 5, and so on, up to  $2n - 2$  and  $2n - 1$ , so 1 is paired to  $2n$ . We deal with this case first, then return to the first one.  $[z, 1]$  is isomorphic to  $NC(n)$ , with the following bijection. Map  $(1, 2n)$  to 1, and each  $(2i, 2i + 1)$  to  $i + 1$  for each  $i$  from 1 to  $n - 1$ .

Then any element in  $[z, 1]$  maps to one of  $NC(n)$ , since any partition of the pairs in  $z$  is a partition of  $n$  elements, and we can show that it must be non-crossing by contradiction. Any crossing partition of  $n$  elements would have some  $a < b < c < d$  where  $a$  and  $c$  are in one block, and  $b$  and  $d$  are in another. If  $a$  is not 1, then an element of  $[z, 1]$  that can map to it must be crossing, which is a contradiction. Likewise if  $a$  is 1. Since each partition of the pairs must correspond to a different partition of the  $n$  elements, it is injective. Every non-crossing partition of  $n$  elements corresponds to an element of  $[z, 1]$ , so it is surjective. The N-atoms of  $[z, 1]$  then correspond to the partitions with one 2-block  $(1, i)$  and the rest being 1-blocks, for each  $i$  from 2 to  $n$ . From lemma 5.11, there is an RA-ordering beginning with these.

Returning to the first case, note that there must be an even number of elements between  $i$  and  $j$ . Now we map each element  $x$  in  $[z, 1]$  to an element  $(f(x), g(x))$  in  $(NC^2(j - i + 1), NC^2(2n - j + i + 1))$  as follows.  $f(x)$  is obtained by deleting elements 1 to  $i - 1$ , and  $j + 1$  to  $n$ .  $g(x)$  is obtained by deleting elements  $i + 1$  to  $j - 1$ . By a similar argument to lemma 5.11, this is bijective. From the conditions on  $i$  and  $j$  above,  $j - i + 1$  and  $2n - j + i + 1$  are both less than  $2n$ . Note that the N-atoms of  $[z, 1]$  are the  $(\alpha, g(z))$  and  $(f(z), \beta)$ , where  $\alpha$  is an N-atom of  $[f(z), 1]$ , and  $\beta$  is an N-atom of  $[g(z), 1]$ . By our induction assumption, there is an RA-ordering of  $[f(z), 1]$  that starts with the N-atoms of  $[f(z), 1]$ , and an RA-ordering of  $[g(z), 1]$  that starts with the N-atoms of  $[g(z), 1]$ . Thus from the proof of lemma 5.7, there is an RA-ordering of  $[z, 1]$  which starts with the N-atoms of  $[z, 1]$ . Hence the result holds.  $\square$

**Definition 5.12.** *The minima joins poset of a join-semilattice  $P$ , or MJ-poset of  $P$ , is the poset that contains the minima of  $P$  along with all the joins of any combination of its minima.*

**Proposition 5.13.** *For any balanced bitstring  $s$ , let  $s'$  be the string formed by deleting each pair of bits in positions  $2k - 1$  and  $2k$  which are different, for any integer  $k$ . Then the MJ-poset of  $T_s$  is isomorphic to the poset  $T_{s'}$ .*

*Proof.* The minima of  $T_s$  all have 1-blocks corresponding to where a pair of bits in positions  $2k - 1$  and  $2k$  are different. If an element of  $T_s$  is the join of some minima, then from Proposition 5.9, it is also the join of those elements in  $NC(n)$ . Therefore, the elements of the MJ-poset of  $T_s$  have 1-blocks in those positions too. Consider any  $s$ -valid partition  $z$  with 1-blocks in those positions. For each block of  $z$  of size greater than 1, each element in it corresponds to a pair of like



bits in the bitstring  $s$ . In the pairing of  $s$  that corresponds to  $z$ , the second bit of the first pair is paired to the first bit of the second pair, and so on. The first bit of the first pair is also paired to the second bit of the last pair. Thus if the first is a pair of 1s, then the second is a pair of 0s, the third is a pair of 1s, and so on. If the first is a pair of 0s, then the second is a pair of 1s, and so on. Also, since the first and last pair must be a pair of 1s and a pair of 0s, or vice versa, the block is in fact of even size.

Let the partition  $a$  have the following form.  $a$  has the 1-blocks of  $z$ . For each block of  $z$  of size greater than 1,  $a$  has 2-blocks which includes the  $(2k - 1)$ th and  $2k$ th element of that block of  $z$ . Likewise, let the partition  $b$  have the 1-blocks of  $z$ . For each block of  $z$  of size greater than 1,  $b$  has 2-blocks which includes the  $2k$ th and  $(2k + 1)$ th element of that block of  $z$ , as well as a 2-block which includes the first and last element of that block of  $z$ . Each 2-block for both  $a$  and  $b$  has one element corresponding to a pair of 1s in  $s$ , and the other corresponding to a pair of 0s, so they are both  $s$ -valid, and hence are minima of  $T_s$ . Both are refinements of  $z$ . Note that from the construction of  $a$  and  $b$ , the join of their restriction to any block of  $z$  is the entire block. Then  $z$  is the join of  $a$  and  $b$ , so  $z$  is in the MJ-poset of  $T_s$ . Note that if we consider the elements apart from those which correspond to where a pair of bits in positions  $2k - 1$  and  $2k$  are different, it corresponds to the bitstring  $s'$ . Thus the elements of the MJ-poset of  $T_s$  are exactly those with 1-blocks corresponding to where a pair of bits in positions  $2k - 1$  and  $2k$  are different, and an  $s'$ -valid partition of the rest. Then each element of the MJ-poset of  $T_s$  can be identified with an element of  $T_{s'}$  by removing the 1-blocks, and vice versa, so the MJ-poset of  $T_s$  is isomorphic to  $T_{s'}$ .  $\square$

**Theorem 5.14.** *Suppose  $T_s$  is isomorphic to the MJ-poset of  $T_s$  and has one maximum. Then  $T'_s$  admits an RA-ordering.*

*Proof.* From Proposition 5.13 we may assume without loss of generality that  $s$  does not have a pair of bits in positions  $2k - 1$  and  $2k$  which are different. Consider its top element,  $z$ . It has blocks of even size, and from Proposition 5.13, the bits corresponding to each element in a block alternate between a pair of 1s and a pair of 0s. Since  $z$  is the only maxima, any element of  $T_s$  must have blocks that are each contained in some block of  $z$ . Number the blocks of  $z$  from 1 to  $k$ . Let the bits corresponding to the  $i$ th block be some bitstring  $t_i$ . Then we can map elements in  $T_s$  to the product of all the  $T_{t_i}$  by letting the  $i$ th coordinate be the partition of the elements in the  $i$ th block of  $z$ . This

map is bijective, so  $T_s$  is isomorphic to the product of the  $T_{t_i}$ .

Now I will show that each  $T_{t_i}$  is isomorphic to some  $NC^2(2l)$ , where  $2l$  is the size of the  $i$ th block of  $z$ . Each element of  $T_{t_i}$  is an element of  $NC^2(2l)$ . Consider any element  $y$  of  $NC^2(2l)$ . Take any block of  $y$ . Since  $y$  is non crossing, any block between the  $j$ th and the  $(j + 1)$ th element of  $y$  is entirely between these, so there are an even number of elements between each of these. Since  $t_i$  comprises of alternating pairs of 1s and pairs of 0s, the  $j$ th and the  $(j + 1)$ th element of  $y$  must correspond to a pair of 1s and a pair of 0s, or vice versa. Thus each block of  $y$  also has bits corresponding to each element in the block alternating between a pair of 1s and a pair of 0s. Then the underlying pairing pairs 1s to 0s, so  $y$  is  $t_i$ -valid. Therefore, the partitions of  $T_{t_i}$  are exactly those of  $NC^2(2l)$ , so they are isomorphic. Let the size of the  $i$ th block of  $z$  be  $l_i$ . Then  $T_s$  is isomorphic to the product of the  $NC^2(2l_i)$ . Each  $NC^2(2l_i)$  admits an RA-ordering from Theorem 5.10. From Theorem 5.6,  $T'_s$  admits an RA-ordering.  $\square$

Now I shall define a weaker version of an RA-ordering, by relaxing one of the requirements.

**Definition 5.15.** *Let  $S$  be a subset of the atoms of a bounded poset  $P$ .  $P$  admits an  $S$ -RA-ordering if its length is 1, or its length is greater than 1 and there is an ordering  $a_1, a_2, \dots, a_n$  of the elements of  $S$  such that:*

- 1) *For all  $j = 1, 2, \dots, n$  the interval  $[a_j, 1]$  admits an  $R$ -RA ordering where  $R$  is the set of atoms of  $[a_j, 1]$  that belong to  $[a_i, 1]$  for some  $i < j$ .*
- 2) *For all  $i < j$ , if  $a_i, a_j < y$  then there is a  $k < j$  and an atom  $z$  of  $[a_j, 1]$  such that  $a_k < z \leq y$ .*

**Definition 5.16.** *A bounded poset  $P$  admits a weak RA-ordering if it admits an  $S$ -RA-ordering, where  $S$  is the set of all the atoms of  $P$ .*

**Proposition 5.17.** *A bounded graded poset that admits an RA-ordering also admits a weak RA-ordering.*

*Proof.* The definition of a weak RA-ordering is weaker than that of an RA-ordering. Using the same order on the atoms of  $P$ , the atoms in the RA-ordering of each interval that appear after those in part one of the definition cannot affect those before, and thus satisfies the first part. The second property of weak RA-orderings is identical to the second property of RA-orderings.  $\square$

I will show that the upper or lower interval of  $T_s$  starting at any point is isomorphic to some  $T_{s'}$ , starting with the following proposition.

**Proposition 5.18.** *Given any  $T_s$  and  $T_t$ , there exists  $T_u$  such that  $T_u$  is isomorphic to the product of  $T_s$  and  $T_t$ .*

*Proof.* Pick a lowest point in the hill diagram for  $s$ . If this is an even number of bits from the left, then rotating the bits on the left of this point to the right would produce a new string  $s'$  so that  $T_{s'}$  is isomorphic to  $T_s$ , and the hill diagram for  $s'$  is never below 0. If there are an odd number of steps, we can add 01 to that point without affecting the structure of the poset, since those bits must then pair to themselves. The new lowest point is thus an even number of bits from the left, so we can do the above, and obtain  $s'$  so that  $T_{s'}$  is isomorphic to  $T_s$ , and the hill diagram for  $s'$  is never below 0. Pick a highest point in the hill diagram for  $t$  and repeat the above argument, obtaining  $t'$  so that  $T_{t'}$  is isomorphic to  $T_t$ , and the hill diagram for  $t'$  is never above 0. Now concatenate  $s'$  and  $t'$  to produce a string  $u$ .

1s represented in the hill diagram as going from height  $x$  to  $x + 1$  must pair to 0s represented as going from height  $x + 1$  to  $x$ , so each pairing occurs only within such height intervals  $[x, x + 1]$ . Thus there can never be a pairing from the  $s'$  part of the string to the  $t'$  part of the string. Any element  $w$  of  $T_u$  corresponds to a pairing of  $u$  which is a pairing of  $s'$  and a pairing of  $t'$ . These correspond to elements  $y$  of  $T_{s'}$  and  $z$  of  $T_{t'}$ , so  $w$  is a concatenation of  $y$  and  $z$ , and can be represented as a pair  $(y, z)$ , where  $y$  is an element of  $T_{s'}$  and  $z$  is an element of  $T_{t'}$ . Note that  $(y, z) \leq (y', z')$  if and only if  $y \leq y'$  and  $z \leq z'$ , so  $T_u$  is isomorphic to the product of  $T_{s'}$  and  $T_{t'}$ , which is isomorphic to the product of  $T_s$  and  $T_t$ . This gives the desired result.  $\square$

**Theorem 5.19.** *For any  $T_s$  and an element  $a$  of  $T_s$ , the lower interval consisting of all elements lesser than or equal to  $a$  is isomorphic to some  $T_{s'}$ . Likewise for the upper interval consisting of all elements greater than or equal to  $a$ .*

*Proof.* Consider the lower interval  $L$  consisting of all elements lesser than or equal to  $a$ .  $a$  is the top element, with blocks  $B_1, B_2, \dots, B_k$ . Let the substrings underlying these blocks be  $s_1, s_2, \dots, s_k$ . There can be no pairing between these substrings in any element of  $L$ , otherwise it would not be less than or equal to  $a$ . Each element of  $L$  consists of a refinement of  $a$ , and thus corresponds to non-crossing pairings of  $s_1, s_2, \dots, s_k$ . On the other hand, any non-crossing pairing of  $s_1, s_2, \dots$

,  $s_k$  corresponds to an element of  $L$ , since it is identified with a non-crossing pairing of  $s$  corresponding to an element of  $T_s$  whose blocks are contained in  $B_1, B_2, \dots, B_k$ . Similarly to the proof of Proposition 5.18, any element of  $L$  can thus be written as  $(x_1, x_2, \dots, x_k)$ , where each  $x_i$  is an element of  $T_{s_i}$ , and  $(x_1, x_2, \dots, x_k)$  is less than or equal to  $(x'_1, x'_2, \dots, x'_k)$  if and only if  $x_i \leq x'_i$  for all  $i$ . Thus  $L$  is isomorphic to the product of all the  $T_{s_i}$ . From Proposition 5.18,  $L$  is isomorphic to some  $T_{s'}$ . For the case of an upper interval, rotation of the string by one bit inverts the whole poset, making it a lower interval. The same result then follows.  $\square$

I will need the following results too.

**Theorem 5.20.** *Let  $P$  be a poset. Suppose  $P'$  admits an  $S$ -RA-ordering, and an upper interval from any point except the bottom is constructible. Let  $U$  be the poset formed from the union of all intervals  $[x, 1]$  for  $x$  in  $S$ . Then  $U$  is constructible.*

*Proof.* I will prove this by induction on the size of  $U$ . The base case is with one element, which is trivial. Now suppose that the assertion is true for posets smaller than  $U$ , and that  $U$  has more than one element. Let the  $S$ -RA-ordering of the elements in  $S$  be  $x_1, x_2, \dots, x_k$ . For any  $j$  from 1 to  $k$ , let  $U_j$  be the union of the intervals  $[x_i, 1]$  for  $i < j$ . Let  $S_j$  be the set of atoms of  $[x_j, 1]$  that belong to  $[x_i, 1]$  for some  $i < j$ . From the definition of  $S$ -RA-ordering,  $[x_j, 1]$  admits an  $S_j$ -RA-ordering. Any point in the intersection of  $[x_i, 1]$  with  $[x_j, 1]$  for  $i < j$  is greater than or equal to some element of  $S_j$  by definition, while any point above some element of  $S_j$  is in some  $[x_i, 1]$  for  $i < j$ . Thus the intersection of  $U_j$  with  $[x_j, 1]$  is the poset  $A$  formed from the union of all intervals  $[y, 1]$  for  $y$  in  $S_j$ , which by the induction assumption is constructible.  $U_2$  is simply  $[x_1, 1]$ , which is constructible. Each  $[x_i, 1]$  is constructible, and the intersection of each  $[x_i, 1]$  and  $U_i$  for  $i$  between 2 and  $k$  is constructible, so by induction  $U$  is constructible. Thus is it true for  $U$  of this size, and by induction it is true for all such  $U$ .  $\square$

This means that if  $P'$  admits a weak RA-ordering, and an upper interval from any point is constructible, then  $P$  is constructible. Another particular case is useful.

**Proposition 5.21.** *Let  $P$  be a join-semilattice, and  $Q$  be its MJ-poset. If  $Q'$  admits a weak RA-ordering, then so does  $P'$ .*

*Proof.* Let  $a$  be the unique maximum of  $Q$ . For any  $x$  in  $Q'$  less than  $a$ , I will show by induction on the size of  $[x, a]$  that if  $[x, a]$  admits

an  $S$ -RA-ordering, so does  $[x, 1]$ . The base case is a coatom of  $Q'$ .  $[x, a]$  admits both an  $\{a\}$ -RA-ordering and a  $\emptyset$ -RA-ordering.  $[x, 1]$  also trivially admits both an  $\{a\}$ -RA-ordering and a  $\emptyset$ -RA-ordering, so the base case is true. Consider some such  $x$  such that it is true for all such  $y$  with the size of  $[y, a]$  less than  $[x, a]$ . If  $[x, a]$  admits an  $S$ -RA-ordering, following a similar argument to the proof of Theorem 5.20 and noticing that the joins of any elements in  $S$  are still in  $Q$ , we see that  $[x, 1]$  admits an  $S$ -RA-ordering too. Thus by induction it is true for all such  $x$ , so  $P'$  admits a weak RA-ordering.  $\square$

Note that the converse is also true, as any weak RA-ordering of  $P'$  involves only joins of the atoms of  $P'$  and is necessarily a weak RA-ordering of  $Q$ .

**Corollary 5.22.** *Let  $P$  be a join-semilattice such that an upper interval from any point is constructible. Let  $Q$  be the MJ-poset of  $P$ . If  $Q'$  has a weak RA-ordering,  $P$  is constructible.*

*Proof.* If  $P$  has a top element, from Proposition 5.21,  $P'$  admits a weak RA-ordering, so  $P$  is constructible from Theorem 5.20. If  $P$  does not have a top element, then by adding a top element and using the same argument,  $P$  with a top element added is constructible. But this is a cone, so  $P$  is constructible.  $\square$

**Proposition 5.23.** *If the MJ-poset of  $T_s$  is equal to  $T_s$  and has more than one maximum, the inverted  $T_s$  cannot be equal to its MJ-poset.*

*Proof.* From Proposition 5.13,  $T_{s'}$  is isomorphic to the MJ-poset of  $T_s$  which is equal to  $T_{s'}$ , where  $s'$  is the string formed from  $s$  by deleting each pair of bits in positions  $2k - 1$  and  $2k$  which are different, for any integer  $k$ . We cannot possibly be changing the structure of  $T_s$  by doing this, thus these bits must be pairing to themselves in all pairings. We can also delete adjacent bits that must pair to each other in every pairing without changing the structure of  $T_{s'}$ , and repeat the procedure. Since the bitstring is finite, it must terminate. Thus we can assume without loss of generality that  $s$  has bits in positions  $2k - 1$  and  $2k$  which are the same, for every integer  $k$ , and that no adjacent bits always pair to each other. Furthermore, we can assume that the bitstring starts with 1s. Let the first 0 be in the  $(2k + 1)$ th position. If we rotate the bit on the left to the right, the poset is inverted, and the first 0 is in the  $2k$ th position. This cannot be pairing to the 1 in the  $(2k - 1)$ th position in all pairings, since we removed all such bits. Let the rotated string be  $t$ . From Proposition 5.13, the MJ-poset of  $T_t$  is isomorphic to  $T_{t'}$  where  $t'$  is the string formed from  $t$  by deleting

each pair of bits in positions  $2k - 1$  and  $2k$  which are different, for any integer  $k$ . Since this removes the  $(2k - 1)$ th and  $2k$ th bits which do not always pair to each other, some pairings are lost. Thus the MJ-poset of  $T_i$  is smaller than  $T_i$  itself, and the result follows.  $\square$

Let  $u$  be a sequence of  $n$  elements such that each element has one of two colours. Let  $B(u)$  be the poset of noncrossing partitions of the elements of  $u$  such that each block only includes elements of a single colour. Let  $A(u)$  be  $B(u)$  with a top element added if there is no unique maximum, and  $B(u)$  if there is. Let the colours be 1 and 2, and let the colour of the first element be 1. Let the colour of an atom of  $A(u)$  be the colour of the elements of its 2-block. Let  $S$  be the set of atoms of  $A(u)$  whose 2-block contains the first element.

**Lemma 5.24.** *For  $n > 1$ ,  $A(u)$  has an  $S$ -RA-ordering.*

*Proof.* Let  $A$  be the subposet of  $A(u)$  consisting of the bottom element, the atoms in  $S$ , and any element of  $A(u)$  greater than an element in  $S$ . A weak RA-ordering of  $A$  then corresponds to an  $S$ -RA-ordering of  $A(u)$ . Let  $B$  be the MJ-poset of  $A$ . The atoms in  $S$  are  $a_2, a_3, \dots, a_k$  where  $a_i$  is the atom whose 2-block contains 1 and the  $i$ th colour 1 element of  $u$ , and  $k$  is the number of colour 1 elements in  $u$ . The join of  $a_{b_1}, a_{b_2}, \dots, a_{b_l}$  is then the partition with an  $l$ -block containing the 1st,  $b_1$ th,  $b_2$ th,  $\dots$ , and  $b_l$ th colour 1 elements, and 1-blocks everywhere else.  $B$  then consists of all those partitions where any block with size greater than 1 must contain the first colour 1 element.

Now map  $B$  to  $NC(k)$  in the following manner. If the block containing the first colour 1 element contains the 1st,  $b_1$ th,  $b_2$ th,  $\dots$ , and  $b_l$ th colour 1 elements, this partition maps to the partition of  $NC(k)$  with a block containing the 1st,  $b_1$ th,  $b_2$ th,  $\dots$ , and  $b_l$ th elements, with 1-blocks everywhere else. Since distinct elements in  $B$  have different blocks containing the first element, this is an injective map. Then  $B$  is isomorphic to a subposet of  $NC(k)$ . The atoms of  $S$  map to those with a 2-block containing the 1st and  $i$ th elements for  $1 < i \leq k$ , with 1-blocks everywhere else. Let this set of atoms be  $S'$ . From the proof of Proposition 5.17 and Lemma 5.11, there is a weak RA-ordering starting with those atoms of  $NC(k)$ . Then there is an  $S'$ -RA-ordering, which corresponds to an  $S'$ -RA-ordering of the image of  $B$  in the map, which corresponds to a weak RA-ordering of  $B$ . Then by Proposition 5.21, this is a weak RA-ordering of  $A$ , which then gives an  $S$ -RA-ordering of  $A(u)$ .  $\square$

Let  $t$  be a balanced bitstring of length  $2n$ . Let  $NC^2(t)$  be the poset of noncrossing partitions of the bits of  $t$  such that each block consists of an even number of alternating 1s and 0s. (ie. either 1010...10 or 0101...01.)

**Theorem 5.25.**  $NC^2(t)$  with a top element admits a weak RA-ordering.

*Proof.* The proof follows a similar argument to that in Theorem 5.10. Let  $NC^2(t)$  with a top element be  $A$ . I will show that  $A'$  admits a weak RA-ordering, and for each atom  $z$  of  $A'$  for  $n > 1$ , there is a  $S_z$ -RA-ordering of the atoms of  $[z, 1]$  where  $S_z$  is the set of N-atoms of  $[z, 1]$ , by induction on  $n$ . The base cases are  $n = 1$  and  $n = 2$ , and are trivially true. Now suppose that the above is true for integers less than  $n$  and greater than one, and I will show that it holds for  $n$  as well.

Again, I will use the atom ordering used in Wachs [9] for partitions with even block size. For each atom of  $A'$ , associate the following word to it. Order each pair in increasing order of its two elements, and list the pairs in lexicographic order. Order the atoms in lexicographic order of the words. Let  $z$  be some atom of  $A'$ , and  $z'$  be an atom that precedes it in the ordering. Suppose their words first differ on the  $k$ th pair. Then the words up to that pair agree. For any word, if  $i$  is the smallest positive integer yet to be listed up to the  $j$ th pair, then the pair containing that must be the next to be listed, so the  $(j + 1)$ th pair consists of  $i$  and some higher element. Thus the  $k$ th pair of both  $z$  and  $z'$  must contain some common element  $a$  and some different higher element. Suppose the  $k$ th pair of  $z$  is  $(a, b)$  and that of  $z'$  is  $(a, b')$ . Since  $z'$  precedes  $z$ ,  $b' < b$ . Let  $t'$  be  $t$  with every bit doubled. (ie. the  $2i - 1$  and  $2i$  bits of  $t'$  is the same as the  $i$  bit in  $t$ .) Note that  $NC^2(t)$  is  $T_{t'}$ , and has joins with an added top element as well as meets with an added bottom element, so  $A'$  is a lattice. Then from Proposition 5.9, if the join of  $z$  and  $z'$  exists in  $NC^2(t)$ , it is their join in  $NC(2n)$ . Suppose it does exist. This join would have one block  $B$  that contains at least  $a$ ,  $b$ , and  $b'$ .

For any pair with an element to the left of  $a$ , it appears in both  $z$  and  $z'$ , so the elements between the two elements of the pair must pair to themselves in both  $z$  and  $z'$ , and likewise for the other elements. Thus an element to the left of  $a$  cannot be in  $B$ . Then the leftmost element of  $B$  is  $a$ . Notice that in either  $z$  or  $z'$ , the elements between any two consecutive elements of  $B$  must pair to themselves, or else there is a block that crosses  $B$ . Let the second leftmost element of  $B$  be  $c$ .  $b$  is not the second leftmost element of  $B$ , and since

it pairs to  $a$  in  $z$ ,  $c$  must pair to another element of  $B$  between  $a$  and  $b$  in  $z$ . Let this be  $d$ . This is to the right of  $c$  and to the left of  $b$ , as  $c$  is the second leftmost element of  $B$ . Let  $z''$  be the partition whose pairs are the same as that of  $z$ , except that instead of the pairs  $(a, b)$  and  $(c, d)$ , it pairs  $a$  with  $c$  and  $d$  with  $b$ . Since the elements between any two consecutive elements of  $B$  must pair to themselves, this is non crossing. If  $a$  is 1,  $c$  is 0,  $d$  is 1, and  $b$  is 0, and the reverse if  $a$  is 0. Thus  $z''$  is an atom of  $A'$ . Since  $c$  precedes  $b$ ,  $z''$  precedes  $z$  in the ordering of atoms.

The join of  $z''$  with  $z$  consists of the other pairings of  $z$  together with the block with elements  $a, c, d, b$ . Each 2-block is in a block of  $z$ , so it is in some block in the join of  $z$  and  $z'$ . The block with elements  $a, c, d, b$  is also in  $B$ , which is in the join of  $z$  and  $z'$ . Then the join of  $z$  and  $z''$  is less than or equal to any element that is greater than or equal to both  $z$  and  $z'$ . This is an atom of  $[z, 1]$ , and  $z''$  precedes  $z$  in the ordering, so this satisfies property two of the weak RA-ordering. Notice that this is an N-atom of  $[z, 1]$ . Then any atom of  $[z, 1]$  belonging to some  $[z', 1]$  for  $z'$  preceding  $z$  is an N-atom.

Now I will show that all N-atoms of  $[z, 1]$  belong to some  $[z'', 1]$  for  $z''$  preceding  $z$ . Pick any N-atom of  $z$ . Then the blocks of that N-atom are the pairs of  $z$  except for one block containing elements  $a < b < c < d$  where  $(a, d)$  and  $(b, c)$  are pairs of  $z$ . Then let  $z''$  be the atom whose pairs are the same as that of  $z$ , except that instead of the pairs  $(a, d)$  and  $(b, c)$ , it pairs  $(a, b)$  and  $(c, d)$ . Then that N-atom belongs to  $z''$ , since  $a$  and  $b$  are different types of elements, and so are  $c$  and  $d$ . Their words first differ in the pairs  $(a, b)$  and  $(a, d)$ , so  $z''$  precedes  $z$ . Thus the atoms of  $[z, 1]$  that are above some atom of  $A'$  that precedes it in the ordering are exactly the N-atoms.

If the join of  $z$  and  $z'$  does not exist in  $NC^2(t)$ , then their join in  $A'$  is the top element. It is above any N-atom, so if  $[z, 1]$  has any N-atoms, the second property of weak RA-orderings is satisfied. I will show this by contradiction. Suppose  $[z, 1]$  has no N-atoms. Without loss of generality, assume that  $a$  is 1. Then  $b$  is 0. There is at least one element between them, since  $b'$  is between them. Consider the element directly to the right of  $a$ , and let this be  $a'$ . Let  $a'$  be paired to  $b''$  in  $z$ . If  $a'$  is 0, then  $a, a', b'', b$  forms a 4-block with alternating 1s and 0s, and does not cross any other block, so it is an N-atom of  $[z, 1]$ , which cannot happen. Then  $a$  is 1. By similar argument, the element to the right of  $b''$  is either  $b$  or is also a 1. Then the elements



from  $a$  to  $b$  look like  $11\dots 01\dots 0 \dots 1\dots 00$ , where each  $\dots$  is a balanced bitstring, and for each  $1\dots 0$ , the 1 pairs to the 0 in  $z$ . By similar argument, each  $1\dots 0$  sequence must also have this structure, or we can find an N-atom. Now consider the bitstring between the first 1 and any 0 other than the last in this sequence. I will show by induction on the length of the string that this cannot be balanced, and has more 1s than 0s. The base case is  $1100$ , with the first 1 pairing to the first 0, and is trivial. Suppose it holds for shorter strings. Suppose that the 0 the first 1 pairs to is in the  $k$ th  $1\dots 0$  sequence. Each  $1\dots 0$  sequence is balanced, so the difference between the number of 1s and the number of 0s is the difference between the number of 1s and the number of 0s in the  $k$ th  $1\dots 0$  sequence up to but not including the 0 in question. By induction assumption, there are more 1s than 0s between the first 1 and that 0. Adding the first 1 retains the conclusion, so it is true for all such sequences. Thus  $a$  cannot pair to  $b'$  between  $a$  and  $b$ , a contradiction. Thus the second property of weak RA-orderings is satisfied in all cases.

By our induction assumption, there is a  $S_z$ -RA-ordering of  $[z, 1]$ . Then the first property holds, and this ordering of the atoms of  $A'$  is a weak RA-ordering.

Now I will show that there is a  $S_z$ -RA-ordering of the atoms of  $[z, 1]$ . Consider  $[z, 1]$ . We have two cases. First, we can find some element  $i$  that pairs to some  $j$  greater than  $i$  in  $z$ , such that their difference is greater than one, and  $(i, j)$  is not  $(1, 2n)$ . Second, this is not true, in which case every element from 2 up to  $2n - 1$  is paired with an adjacent element. If 2 is paired with 1, then 3 is paired with 4, and so on, up to  $2n - 1$  and  $2n$ . Then there are no N-atoms and the assertion that there is a  $S_z$ -RA-ordering of the atoms of  $[z, 1]$  is trivially true. If 2 is paired to 3, then 4 is paired to 5, and so on, up to  $2n - 2$  and  $2n - 1$ , so 1 is paired to  $2n$ . We deal with this case first, then return to the first one. Let  $u$  be an  $n$  element string with two colours such that the first element is colour 1, and the  $i$ th element for  $i > 1$  is colour 1 if the  $(2i - 2)$ th bit of  $t$  is different from the 1st bit, and colour 2 if it is the same. Let  $A(u)$  be as in Lemma 5.24 Map  $[z, 1]$  to  $A(u)$  as follows. 1 goes to 1. Otherwise, map  $(1, 2n)$  to 1, and each  $(2i, 2i + 1)$  to  $i + 1$  for each  $i$  from 1 to  $n - 1$ . Any partition of the pairs in  $z$  is a partition of  $n$  elements, and is noncrossing from the proof of Theorem 5.10. Any block of pairs in  $[z, 1]$  is of the same colour when mapped to  $A(u)$ . Then any element in  $[z, 1]$  maps to one of  $A(u)$ . Since each partition of the pairs must correspond to a different partition of

the  $n$  elements, it is injective. Every partition of  $A(u)$  is mapped to by an element of  $[z, 1]$ , as a block of the same colour corresponds to a block of alternating 1s and 0s in  $NC^2(t)'$ , so it is surjective. Therefore  $[z, 1]$  is isomorphic to  $A(u)$ . The N-atoms of  $[z, 1]$  then map to the partitions with one 2-block  $(1, i)$  and the rest being 1-blocks, for each  $i$  from 2 to  $n$  such that  $i$  is of colour 1. From lemma 5.24, there is an RA-ordering beginning with these.

Returning to the first case, note that there must be an even number of elements between  $i$  and  $j$ . Now we map  $[z, 1]$  to  $(NC^2(t'), NC^2(t''))$  with a top element added, as follows.  $t'$  is obtained from  $t$  by deleting elements 1 to  $i - 1$ , and  $j + 1$  to  $n$ .  $t''$  is obtained from  $t$  by deleting elements  $i + 1$  to  $j - 1$ . 1 maps to 1. For  $x$  not 1 in  $[z, 1]$ , it maps to  $(f(x), g(x))$ , where  $f(x)$  is obtained from  $x$  by deleting elements 1 to  $i - 1$ , and  $j + 1$  to  $n$ .  $g(x)$  is obtained from  $x$  by deleting elements  $i + 1$  to  $j - 1$ . By a similar argument to lemma 5.11, this is bijective. From the conditions on  $i$  and  $j$  above,  $j - i + 1$  and  $2n - j + i + 1$  are both less than  $2n$ . Note that the N-atoms of  $[z, 1]$  are the  $(\alpha, g(z))$  and  $(f(z), \beta)$ , where  $\alpha$  is an N-atom of  $[f(z), 1]$ , and  $\beta$  is an N-atom of  $[g(z), 1]$ . By our induction assumption, there is a  $S_{f(z)}$ -RA-ordering of  $[f(z), 1]$ , and a  $S_{g(z)}$ -RA-ordering of  $[g(z), 1]$ . Following the proof of Lemma 5.8, there is a  $S_z$ -RA-ordering of  $[z, 1]$ , as the ordering extends over such a product of posets. Hence the result holds.  $\square$

**Theorem 5.26.** *Let  $T_s$  be isomorphic to the MJ-poset of  $T_s$ . Then  $T'_s$  with an added top element admits a weak RA-ordering.*

*Proof.* From Proposition 5.13, we may assume that  $s$  consists of a concatenation of 11 and 00 sequences. Any element in  $T_s$  must have blocks whose underlying string consists of alternating 11 and 00 pairs, otherwise a 0 would pair to a 0 or a 1 would pair to a 1 in the underlying pairing. Likewise, any such partition is  $s$ -valid and is then a member of  $T_s$ . Thus  $T_s$  has exactly the form in Theorem 5.25, so  $T'_s$  with an added top element admits a weak RA-ordering.  $\square$

Now I come to the main theorem of this section.

**Theorem 5.27.**  *$T_s$  is constructible.*

*Proof.* I will show this by induction on the size of  $T_s$ . The base case is one element, and is trivial. Notice that since  $T_s$  is convex in  $NC(n)$ , if  $T_s$  has one minimum and one maximum, then it is isomorphic to an interval of  $NC(n)$ . Such a  $T_s$  would then have an RA-ordering, since intervals in  $NC(n)$  have an RA-ordering. Then  $T_s$  is CL-shellable [9],

and hence is constructible.

Suppose that  $T_s$  has more than one minimum. Then all upper intervals beginning at some point in  $T_s$  is of a smaller size than  $T_s$ . From Theorem 5.19, it is isomorphic to some  $T_{s'}$ , and by induction assumption, it is constructible. Now further suppose that the MJ-poset  $A$  of  $T_s$  has a unique maximum. From Proposition 5.13 and Theorem 5.14,  $A'$  admits an RA-ordering. From Proposition 5.17,  $A'$  admits a weak RA-ordering. Let  $T$  be  $T_s$  if it has a top element, and  $T_s$  with a top element added otherwise. Since  $A$  has a unique maximum, adding a top element to  $T_s$  would not add any elements to its MJ-poset. So the MJ-poset of  $T_s$  is still  $A$ . From Proposition 5.21,  $T'$  admits a weak RA-ordering. Since cones of constructible posets are constructible, upper intervals from any point of  $T'$  are also constructible. From Theorem 5.20,  $T$  is constructible. Then  $T_s$  is constructible.

If on the other hand  $A$  has more than one maximum, let  $B$  be  $A$  with a top element added.  $T_s$  cannot then have a top element, so  $T$  has a top element added. Then  $B$  is the MJ-poset of  $T$ . From Theorem 5.26,  $B'$  admits a weak RA-ordering. Then from Proposition 5.21,  $T'$  admits a weak RA-ordering. Following the above,  $T_s$  is constructible.

The remaining case is where  $T_s$  has one minimum and more than one maximum. Inverting  $T_s$  gives us  $T_{s'}$ , where  $s'$  is  $s$  with one bit rotated. This then has more than one minimum, and from the above, it is constructible. Inverting the poset does not change its structure, thus  $T_s$  is constructible. Thus the result holds.  $\square$

**Conjecture 5.28.**  $T_s$  is contractible.

A contractible complex can be built in much the same recursive manner as in Definition 5.1, except that the base case is different. Since the base case is true in  $T_s$ , the above approach should work to prove contractibility as well. Note that from Proposition 3.12, this is true when  $T_s$  is graded with length 1.

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