

# Circulant networks of identical Kuramoto oscillators: Seeking the densest network that does not globally synchronize and the sparsest one that does

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There is a critical connectivity  $0 < \mu_c < 1$  for systems of identical Kuramoto oscillators. Any network of size  $n$  in which each oscillator interacts with at least  $\mu(n-1)$  others is globally synchronizing if  $\mu \geq \mu_c$ ; otherwise, it may not be. The best known bounds on  $\mu_c$  are  $15/22 \approx 0.6818 \leq \mu_c \leq (3 - \sqrt{2})/2 \approx 0.7929$ . Focusing on circulant networks leads us to conjecture that  $\mu_c = 0.75$ . At the sparse end of the connectivity spectrum, we find that a ring of  $n$  oscillators can be turned into a globally synchronizing network by adding just  $\mathcal{O}(n \log_2 n)$  edges.

Coupled nonlinear oscillators often fall into sync spontaneously. Examples range from flashing fireflies and neural populations to arrays of Josephson junctions and nanoelectromechanical oscillators [1–7]. On the theoretical side, many researchers have explored how the tendency to synchronize is affected by network structure [8–20]. This is the topic of the present Letter.

We say that a system of coupled oscillators globally synchronizes if it converges to a state with all the oscillators in phase, starting from all initial conditions except a set of measure zero. Intuitively, one expects that dense networks should favor synchronization, whereas sparse networks might support waves and other patterns besides synchrony. In 2012, Taylor [11] explored these issues using a system of identical Kuramoto oscillators. He proved that if each oscillator is coupled to at least 93.95% of the others, the system synchronizes; hence  $\mu_c \leq 0.9395$ . The striking thing about Taylor’s result is that the network could be arbitrary in all other respects (random, regular, or in between). Recently, Ling, Xu, and Bandeira [20] refined Taylor’s argument to show that 79.29% connectivity ensures global synchronization; thus  $\mu_c \leq 0.7929$ . On the other hand, competing attractors called twisted states can coexist with the synchronous state for certain networks whose connectivity is less than 68.18% [9, 15]. Thus  $0.6818 \leq \mu_c \leq 0.7929$ .

In this Letter, we use symmetric structures called circulant networks to sharpen the understanding of  $\mu_c$ . We construct a family of networks (Fig. 1, top row) whose connectivity approaches 75%, and which lie on the razor’s edge of synchrony: along with the stable synchronous state, they have a competing, nearly stable, twisted state. In each case, that twisted state has all negative eigenvalues, apart from four zero eigenvalues which render linear analysis insufficient. Simulations show that, sadly, this twisted state is weakly unstable. Thus, any future attempt to show  $\mu_c < 0.75$  must contend with this sequence of networks. Its existence leads us to conjecture that  $\mu_c = 0.75$ . There is also a surprise at the opposite end of the connectivity spectrum (Fig. 1, bottom row). Merely connecting each oscillator to a logarithmically small number of neighbors suffices to destabilize all

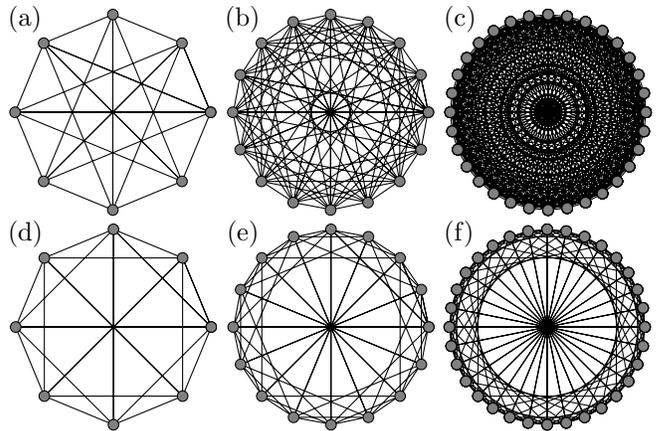


FIG. 1. Dense circulant networks and sparse ones of size  $n = 8, 16,$  and  $32$ . Panels (a)-(c) show networks with a twisted state whose associated Jacobian matrices have all eigenvalues  $\leq 0$ . These networks are tantalizingly close to being globally synchronizing. We wonder if they could be tweaked to be truly so. Panels (d)-(f) show sparse circulant networks that have no stable twisted states. These are promising candidates for the sparsest circulant networks that globally synchronize, but we have not ruled out the existence of other attractors.

the twisted states of a ring, thereby converting it (we conjecture) into a globally synchronizing network.

To find these example networks, we consider a homogeneous Kuramoto model in which each oscillator has the same natural frequency (which can be set to zero without loss of generality, by going into a suitable rotating frame). The governing equations are

$$\frac{d\theta_j}{dt} = \sum_{k=0}^{n-1} A_{jk} \sin(\theta_k - \theta_j), \quad 0 \leq j \leq n-1. \quad (1)$$

Here  $A_{jk} = A_{kj} = 1$  if oscillator  $j$  is coupled to oscillator  $k$ ; thus, all interactions are symmetric, equally attractive, and normalized to unit strength. Since the adjacency matrix  $A$  is symmetric, (1) is a gradient system [8, 9]. Thus all the attractors are equilibrium points, so to analyze the long-term global dynamics of (1), it suffices to analyze the local stability of equilibria.

Circulant networks turn out to be a rich source of

graphs for our purposes. A circulant network is a graph whose adjacency matrix  $A$  has constant diagonals, such that each row is a circularly shifted version of the preceding row, i.e.,

$$A = \begin{bmatrix} a_0 & a_1 & \cdots & a_2 & a_1 \\ a_1 & a_0 & a_1 & & a_2 \\ \vdots & a_1 & a_0 & \ddots & \vdots \\ a_2 & & \ddots & \ddots & a_1 \\ a_1 & a_2 & \cdots & a_1 & a_0 \end{bmatrix}.$$

The matrix  $A$  is prescribed by selecting the values of  $a_0, a_1, \dots, a_{\lfloor n/2 \rfloor}$  as either 0 or 1. We assume throughout that  $a_0 = 0$  so that no oscillator is coupled to itself.

The eigenvalues of any circulant matrix are known analytically [21] and are given by

$$\lambda_p(A) = \sum_{s=1}^{n-1} a_s e^{2\pi i p s / n}, \quad 0 \leq p \leq n-1, \quad (2)$$

where  $a_s = a_{n-s}$  for  $\lfloor n/2 \rfloor < s < n$ . The reflectional symmetry  $a_s = a_{n-s}$  implies that the eigenvalues in (2) are real and  $\lambda_p(A) = \lambda_{n-p}(A)$  for  $1 \leq p \leq n-1$ . Moreover, the eigenvectors of  $A$  are given by [21]

$$\begin{aligned} A \underline{v}_p &= \lambda_p(A) \underline{v}_p, & 0 \leq p \leq \lfloor n/2 \rfloor, \\ A \underline{w}_p &= \lambda_p(A) \underline{w}_p, & 1 \leq p \leq \lfloor n/2 \rfloor - 1, \end{aligned} \quad (3)$$

where

$$\underline{v}_p = \begin{bmatrix} 1 \\ \cos\left(\frac{2\pi p}{n}\right) \\ \vdots \\ \cos\left(\frac{2\pi p(n-1)}{n}\right) \end{bmatrix}, \quad \underline{w}_p = \begin{bmatrix} 0 \\ \sin\left(\frac{2\pi p}{n}\right) \\ \vdots \\ \sin\left(\frac{2\pi p(n-1)}{n}\right) \end{bmatrix}.$$

Several equilibrium points of (1) can be derived from linear algebra when  $A$  is a circulant matrix. Let  $\underline{\theta} = (\theta_0, \dots, \theta_{n-1})^\top$ ,  $\underline{c} = \cos(\underline{\theta})^\top$ , and  $\underline{s} = \sin(\underline{\theta})^\top$ , where the superscript  $\top$  denotes the vector transpose. Then since  $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$ , a vector  $\underline{\theta}$  is an equilibrium point of (1) if and only if

$$D_{\underline{c}} A \underline{s} = D_{\underline{s}} A \underline{c}, \quad D_{\underline{c}} = \text{diag}(\underline{c}), \quad D_{\underline{s}} = \text{diag}(\underline{s}),$$

where  $\text{diag}(\underline{c})$  is a diagonal matrix with  $\underline{c}$  on the diagonal. Due to the structure of (1), if  $\underline{\theta}$  is an equilibrium point, then it remains an equilibrium point if all its arguments are shifted by the same angle  $\Theta$ . To avoid these trivial rotations, we set  $\theta_0 = 0$ . Under this restriction, we find from (3) that

$$\underline{\theta}^{(p)} = \left( 0, \frac{2\pi p}{n}, \dots, \frac{2\pi p(n-1)}{n} \right)^\top \quad (4)$$

is an equilibrium point of (1) for any  $0 \leq p \leq \lfloor n/2 \rfloor$ . When  $p = 0$ ,  $\underline{\theta}^{(0)} = (0, \dots, 0)^\top$  and we call this the

synchronous state, corresponding to all the oscillators being in phase. We call (4) a twisted state for  $1 \leq p \leq \lfloor n/2 \rfloor$  as such a state physically corresponds to arranging the oscillators so that their phases differ by a constant amount from one oscillator to the next, twisting uniformly through  $p$  full revolutions as we circulate once around the network. For any circulant network, the twisted states are always equilibrium points but there are usually other equilibria too [15].

To investigate the stability of the twisted states, we take a look at the Jacobian associated to (1). It is given by, for  $0 \leq p \leq \lfloor n/2 \rfloor$ ,

$$(J_p)_{jk} = \begin{cases} A_{jk} \cos(\theta_k^{(p)} - \theta_j^{(p)}), & j \neq k, \\ -\sum_{s=0}^{n-1} A_{js} \cos(\theta_s^{(p)} - \theta_j^{(p)}), & j = k. \end{cases}$$

It can be verified that  $J_p$  is a symmetric circulant matrix for  $0 \leq p \leq \lfloor n/2 \rfloor$ . This means that the eigenvalues of  $J_p$  at the equilibrium points in (4) can be analytically derived. For any  $0 \leq p \leq \lfloor n/2 \rfloor$ , we have [21]

$$\lambda_r(J_p) = \sum_{s=1}^{n-1} a_s \cos(pt_s) [-1 + \cos(rt_s)], \quad (5)$$

where  $t_s = 2\pi s/n$ . The stability of the twisted state (4) depends on the signs of  $\lambda_0(J_p), \dots, \lambda_{n-1}(J_p)$ . If one of these numbers is positive, then  $\underline{\theta}^{(p)}$  is unstable. The matrix  $J_p$  will always have at least one zero eigenvalue because of the trivial rotations by  $\Theta$  mentioned earlier; if all other eigenvalues are negative, then  $\underline{\theta}^{(p)}$  is stable.

To illustrate the general theory, we begin by applying it to two familiar examples of circulant networks.

*Ring network.* Consider a ring of oscillators with nearest-neighbor coupling so that  $a_1 = 1$  and  $a_k = 0$  for  $2 \leq k \leq \lfloor n/2 \rfloor$ . We find that

$$\lambda_r(J_p) = 2 \cos\left(\frac{2\pi p}{n}\right) \left[ -1 + \cos\left(\frac{2\pi r}{n}\right) \right].$$

Hence  $\underline{\theta}^{(p)}$  is unstable if and only if the twist  $p$  satisfies  $n/4 < p \leq \lfloor n/2 \rfloor$ .

*The Wiley–Strogatz–Girvan (WSG) network* [9]. This is the network with adjacency matrix given by  $a_1 = \dots = a_\ell = 1$ , where  $1 \leq \ell < \lfloor n/2 \rfloor$ . In other words, each oscillator on a ring is connected to its  $\ell$  nearest neighbors on either side. Trigonometric identities yield

$$\begin{aligned} \lambda_r(J_p) &= 2 \sum_{s=1}^{\ell} \cos\left(\frac{2\pi p s}{n}\right) \left[ -1 + \cos\left(\frac{2\pi r s}{n}\right) \right] \\ &= \frac{1}{2} U_{2\ell}(x_{p+r}) + \frac{1}{2} U_{2\ell}(x_{p-r}) - U_{2\ell}(x_p), \end{aligned}$$

where  $U_{2\ell}$  is the degree  $2\ell$  Chebyshev polynomial of the second kind [22, Table 18.3.1] and  $x_p = \cos(p\pi/n)$ . From this formula, we can numerically verify that there

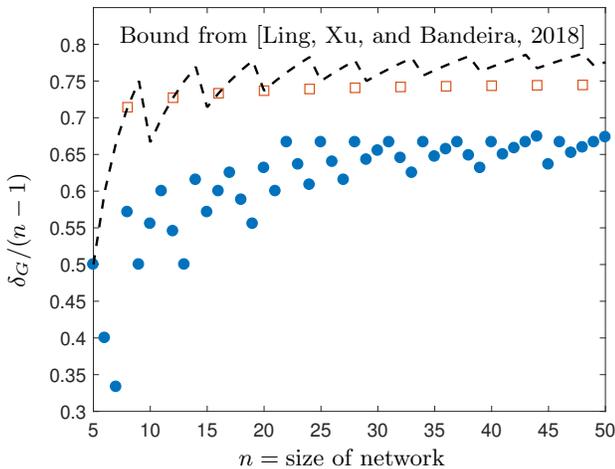


FIG. 2. Connectivity of the densest circulant networks with a twisted state that is either stable or weakly unstable for  $5 \leq n \leq 50$ . The blue dots indicate networks with at least one stable twisted state. (All the nontrivial eigenvalues are negative.) The red dots show a sequence of dense circulant networks with a twisted state that has all negative eigenvalues apart from 4 zero eigenvalues. The dashed line is a rigorous bound for each  $n$ , above which no stable states (of any kind) exist apart from synchrony [20, Thm. 3.1].

is a sequence of networks with connectivity tending to 68.09% for which the 1-twist is a stable equilibrium. This matches the results derived in [9]. Canale and Monzón formed a sequence of networks from the lexicographic product of a WSG network together with complete graphs to show that  $\mu_c \geq 15/22 \approx 0.6818$  [15].

Having reviewed circulant networks, we now use them to look for dense networks that may not synchronize. Every vertex in a circulant network  $G$  has the same degree,  $\delta_G$ , so an equivalent aim is to maximize  $\delta_G/(n-1)$  over all networks that fail to globally synchronize. To do this, we conducted a numerical search over all circulant networks of size  $5 \leq n \leq 50$  having at least one stable twisted state (for which all nontrivial eigenvalues are negative). The blue dots of Fig. 2 show the maximum value of  $\delta_G/(n-1)$  for each  $n$ . They all lie below the best known lower bound of  $15/22 \approx 0.6818$  [15]. But, notice the red squares that are far above the blue dots. These represent networks perched on the razor’s edge of synchrony. This sequence of remarkable networks squeezes up to 75% connectivity as  $n \rightarrow \infty$ . Half of these networks have size  $n = 8m$  and the other half have size  $n = 8m + 4$ . (The stability properties for these two cases are similar so from now on we restrict our attention to  $n = 8m$ .)

For  $n = 8m$ , consider the circulant network  $G$  such that  $a_j = 1$  if and only if  $\text{mod}(j, 4) \neq 2$  for  $1 \leq j \leq 4m$ . The degree of each vertex is  $\delta_G = 6m - 1$ . The twisted state  $\underline{\theta}^{(2m)}$  in (4) has no positive eigenvalues for this network. To see this, use (5) and the identity  $-1 +$

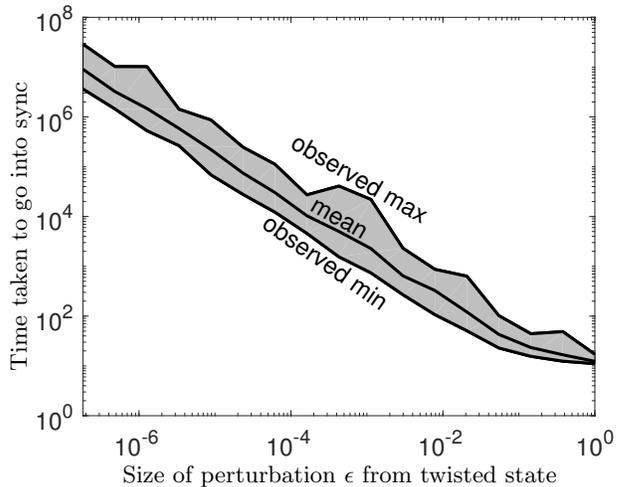


FIG. 3. Slow escape from a weakly unstable twisted state for  $n = 8$ . We integrated (1) starting from random initial conditions within an Euclidean ball of size  $\epsilon$  about the twisted state (4) for  $p = 2$ . For each  $\epsilon$ , we repeated the simulation 100 times, to gather statistics, and computed the time taken to reach the in-phase state to within machine precision.

$\cos(2x) = -2 \sin^2(x)$  to obtain

$$\begin{aligned} \lambda_r(J_{2m}) &= -2 \sum_{s=1}^{2m-1} \cos(2\pi s) \sin^2\left(\frac{\pi sr}{2m}\right) \\ &\quad - 2 \sum_{s=0}^{2m-1} \cos(2\pi s + \pi/2) \sin^2\left(\frac{\pi(4s+1)r}{8m}\right) \\ &\quad - 2 \sum_{s=0}^{2m-1} \cos(2\pi s + 3\pi/2) \sin^2\left(\frac{\pi(4s+3)r}{8m}\right). \end{aligned}$$

Noting that  $\cos(2\pi s + \pi/2) = \cos(2\pi s + 3\pi/2) = 0$  and  $\cos(2\pi s) = 1$  for any integer  $s$ , we conclude that

$$\lambda_r(J_{2m}) = -2 \sum_{s=1}^{2m-1} \sin^2\left(\frac{\pi sr}{2m}\right) \leq 0, \quad 0 \leq r \leq n-1.$$

This means that  $J_{2m}$  has all negative eigenvalues apart from four that vanish (corresponding to  $r = 0, 2m, 4m, 6m$ ), as promised earlier. We have found a dense network of size  $8m$  where the stability or instability of the  $2m$ -twist cannot be determined by the eigenvalues of the Jacobian. Unfortunately, these twisted states turn out to be weakly (nonlinearly) unstable; otherwise, this sequence would have dramatically improved the lower bound on  $\mu_c$  from 0.6818 to 0.75.

Figure 3 shows the consequences of this weak instability. For initial conditions starting within an  $\epsilon$ -neighborhood of a twisted state, it takes an algebraically long time of  $\mathcal{O}(\epsilon^{-1})$  to escape that neighborhood before settling into sync. This super-slow escape time hints that these networks are poised at criticality. Perhaps, just a

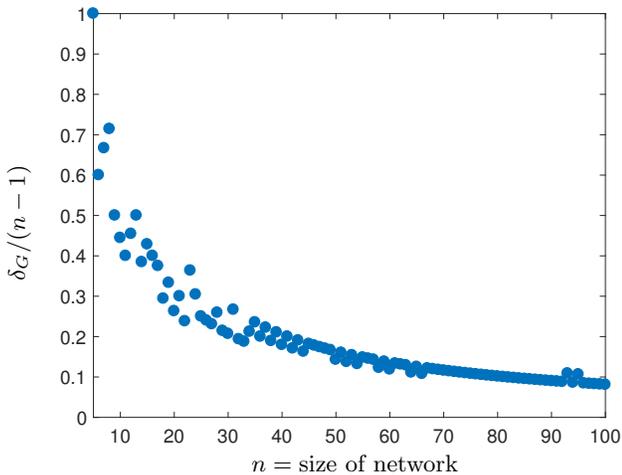


FIG. 4. Connectivity of the sparsest circulant networks for which all the twisted states are unstable. The trend observed here led us to a particular sequence of sparse circulant networks, all of which appear to be globally synchronizing.

small tweak — in the network topology, the sine coupling function, or the edge weights — would be enough to nudge the system towards being globally synchronizing. In any case, this leads us to conjecture that  $\mu_c = 0.75$ .

As a counterpoint to the search for dense networks that fail to synchronize, we have also looked for sparse networks that do globally synchronize. Figure 4 shows, for  $5 \leq n \leq 100$ , the sparsest circulant networks that lack any stable twisted state as revealed by a brute-force numerical search. The trend in Fig. 4 is  $\delta_G = \mathcal{O}(\log_2 n)$ . Hence, the connectivity  $\delta_G/(n-1)$  of these networks tends to zero as  $n \rightarrow \infty$ .

To understand this logarithmic trend, consider the circulant network of size  $n = 2^m$ , where  $a_j = 1$  for all powers of 2 less than or equal to  $2^{m-1}$  (Fig. 1, bottom row). The other  $a_j$ 's are zero, or determined by the symmetry condition  $a_s = a_{n-s}$  for  $1 \leq s \leq n/2$ . This network is constructed by connecting a logarithmically small number of neighbors to each vertex of a ring. The final network has a total of  $(2 \log_2 n - 1)n/2$  edges.

All the twisted states  $\underline{\theta}^{(1)}, \dots, \underline{\theta}^{(n-1)}$  are unstable for this network. To prove this, we need to show that each Jacobian  $J_1, \dots, J_{n/2}$  has at least one strictly positive eigenvalue. Consider the  $p$ th eigenvalue of  $J_p$ . By (5) and the fact that  $\cos(x)[-1 + \cos(x)] = [1 - 2\cos(x) + \cos(2x)]/2$ , we find that

$$\lambda_p(J_p) = m + \sum_{k=0}^{m-2} \cos\left(\frac{2\pi p 2^{k+1}}{2^m}\right) - 2 \sum_{k=0}^{m-2} \cos\left(\frac{2\pi p 2^k}{2^m}\right) - \cos(p\pi).$$

This expression can be simplified to

$$\lambda_p(J_p) = m - \sum_{k=1}^{m-2} \cos\left(\frac{2\pi p 2^k}{2^m}\right) - 2 \cos\left(\frac{2\pi p}{2^m}\right) \quad (6)$$

$$\geq m - (m-2) - 2 = 0,$$

where we can only hope to have an equality in (6) if  $\cos(\frac{2\pi p}{2^m}) = 1$ , i.e., if  $2^{-m}p$  is an integer. Since  $0 \leq p \leq 2^m - 1$ , we conclude that  $\cos(\frac{2\pi p}{2^m}) < 1$  for  $1 \leq p \leq 2^m - 1$ . In other words,  $\lambda_p(J_p) > 0$  for  $1 \leq p \leq 2^m - 1$ . Thus, all the twisted states are unstable.

The network we have just constructed is very sparse. We conjecture that if a circulant network is even more sparse, namely  $\delta_G \ll \log_2 n$  as  $n \rightarrow \infty$ , then at least one of the twisted states must become stable. We go further and conjecture that our sparse circulant examples not only lack stable twisted states; they lack competing attractors of any kind. Proving this conjecture for all  $n = 2^m$  seems difficult because it requires checking that the synchrony is the only stable state of (1). The conjecture holds for  $n = 2$  and  $n = 4$  because then the networks are complete graphs, which are known to be globally synchronizing [5]. For  $n = 8$ , we used the computational algebraic geometry package *Macaulay2* [23] to confirm that there are 262 real isolated equilibrium points of (1) and all these equilibria are unstable except for the synchronous state. (There are an additional 1,008 isolated complex-valued equilibria, and also 5 continuous families of unstable equilibria.) For larger  $n$ , the number of real equilibria grows exponentially. For  $n = 16$ , there are at least 32,768 real equilibria, but there could be as many as 1,073,741,824 complex-valued ones. The lesson is that a brute-force search quickly becomes computationally hopeless. A new idea is needed.

Despite the unfinished nature of the theory, the study of global synchronization can have practical implications. Existing theorems for a different class of oscillators (pulse-coupled oscillators) [4] have been applied to sensor networks and distributed clock synchronization [24–27], and to the design and implementation of ultra-low-power radio systems [28] with potential uses that include intrusion detection, keeping track of workers in coal mines, personal health care, and automated drug-delivery [25]. In the same vein, the results presented here for Kuramoto oscillators may open up additional ways to build useful networks that spontaneously synchronize themselves.

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