A REVIEW OF "ORTHOGONAL POLYNOMIALS IN MATLAB: EXERCISES AND SOLUTIONS" BY GAUTSCHI

ALEX TOWNSEND*

Walter Gautschi is a leading expert in constructive orthogonal polynomials and special functions. He is well-known to the SIAM readership through his fundamental contributions to numerical analysis that span over 65 years. So, you can imagine my excitement when his new book "Orthogonal Polynomials in MATLAB: Exercises and Solutions" arrived in the mail. I hastily ripped off the plastic packaging and blocked out the following weekend to devour its words of wisdom. Before finishing breakfast that Saturday, I realized that this is not an armchair book for a weekend. A desk chair, MATLAB, and the manuscript [2] are useful accessories.

Several months later I picked up the book again with those accessories. I was pleased I did. Gautschi reminds us of some important concepts and computational techniques in constructive orthogonal polynomials. To get the most out of this book, I had to execute some MATLAB codes, regularly consult [2], and carefully work through some of the derivations. I suspect you will need to do the same.

Since every set of orthogonal polynomials satisfies a three-term recurrence relationship, Gautschi views the recurrence relation as the definitive characterization of a set of orthogonal polynomials. As he describes, it provides access to Gauss quadrature rules (Chap. 4), evaluation procedures (Sec. 5.2), and the zeros of orthogonal polynomials (Sec. 3.3). Thus, the goal of Chapter 2 (which takes up a third of the book) is to compute the recurrence coefficients for the associated set of orthogonal polynomials when given an alternative description.

There is the discretized Stieltjes procedure (Sec. 2.5) for computing the recurrence when the inner-product defining the orthogonality is known, the Chebyshev algorithm when the first $2n$ moments are given (Sec. 2.4), the modified Chebyshev algorithm when modified moments are prescribed, and a modification algorithm when a known weight is updated by a rational function (Sec. 2.6). These algorithms are sophisticated and state-of-the-art. Most are numerical procedures, but the ill-conditioning inherent in moments means that the Chebyshev algorithm is usually performed in symbolic mode. I appreciated the mastery in Chapter 2 and the distinction between problems that can be solved numerically and those that require symbolic manipulations.

Chapter 2 also reminds us why folklore is abound in the subject of constructive orthogonal polynomials. Algorithmic details do matter! As a beginner it is easy to be mislead by quick experiments. After experience, one can successfully identity how to modify a question to a well-conditioned problem and carefully tip-toe around nearby numerical pitfalls. To get that experience, you will need that desk chair. Decades after Gautschi's original work on the subject in Chapter 2, this material is still important and very welcomed.

Chapter 2 pays off in chapter 4 on Gauss quadrature, where the Golub–Welsch algorithm is the main computational tool [3]. While the key relationship between the eigenvalues of a Jacobi matrix and the Gauss quadrature nodes is unceremoniously stated in Exercise 4.3, Gautschi beautifully illustrates the so-called *circle theorem*, i.e., appropriately scaled Gauss quadrature weights asymptotically tend to a semicircle. He goes on to present many standard and nonstandard Gauss quadrature

*Department of Mathematics, Cornell University, Ithaca, NY 14850. (townsend@cornell.edu)

rules, including Gauss–Kronrod quadrature (used in adaptive numerical integration packages, see the `quadgk` command in MATLAB [4]) and Gauss–Turán quadrature. Chapter 4 will be of interest to those readers searching for a collection of theoretical properties of Gauss quadrature rules. I will not hide my disappointment that in 54 pages on Gauss quadrature the recent advances in computing the nodes and weights were not mentioned [5]. This is probably because these new algorithms do not directly exploit the three-term recurrence relation.

The last chapter is concerned with the approximation techniques that can be devised from orthogonal polynomials. I was most intrigued by the spline approximation techniques in Section 5.3 that preserve the moments of functions. It is shown that with a regularity assumption on a function, Gauss quadrature can be employed to compute coefficients in a moment-preserving spline approximation. I imagine this could be useful for practitioners in probability theory that want to construct an approximation to a probability distribution function.

The most unusual feature of the book is its style. I greatly struggled with it. Except for the first chapter and appendices, the chapters are essentially a list of exercises with worked solutions. Gautschi asks an exam-style question on a topic, expertly answers it, and then asks another question. This is repeated for about 250 pages. This makes the book useful for a student or instructor that wants a list of worked exam-like questions on topics such as orthogonal polynomials, Gauss quadrature, or least squares approximation. However, I do not think it is ideal for someone learning the material. At times I found it difficult to understand the motivation behind an exercise and regularly consulted [2] to brush up on some details before fully appreciating the elegant solutions. I suspect that beginners will be better served by reading Gautschi's 2004 book [1], which covers similar material.

Accompanying the book are two software packages called OPQ (Orthogonal Polynomials and Quadrature) and SOPQ (Symbolic OPQ). The appendices list the commands in these packages and a brief description of each one. Users of those two software packages will find that this book nicely serves as documentation. There is even a convenient software index in addition to the standard topics index. As I was happily experimenting with OPQ, I found the software index complete and very useful. The publicly available codes in OPQ and SOPQ are a great service to the orthogonal polynomial community.

If you are someone that is interested in constructive orthogonal polynomials, but has not yet read Gautschi's 2004 book [1], then I recommend you read that first. In my view this new 2016 book is a good source of exercises for practice, homework, and examinations, but the style makes it difficult to read linearly and awkward to learn from. Nevertheless, a welcomed addition to the literature on a beautiful subject.

REFERENCES

[1] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, Oxford University Press, 2004.

[2] W. Gautschi, *Orthogonal polynomials, quadrature, and approximation: computational methods and software (in MATLAB)*, Orthogonal polynomials and special functions, Springer Berlin Heidelberg, (2006), pp. 1–77.

[3] G. H. Golub and J. H. Welsch, *Calculation of Gauss quadrature rules*, Math. Comput., 23 (1969), pp. 221–230.

[4] L. F. Shampine, *Vectorized adaptive quadrature in MATLAB*, J. Comput. Appl. Math., 211 (2008), pp. 131–140.

[5] A. Townsend, *The race for high order Gauss–Legendre quadrature*, SIAM News, March, 2015.