Chebfun2: An extension of Chebfun to two dimensions

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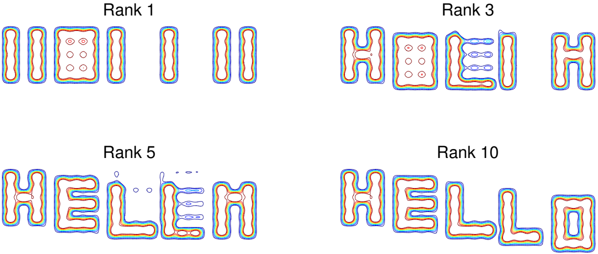
What is the Chebfun2 project?

- **Project:** An attempt to generalise Chebfun to rectangles.
- **Mathematical idea:** A continuous analogue of low-rank matrices and iterative Gaussian elimination.
- **Software:** 12,000 lines of MATLAB code and 203 m-files.



The low rank approximation of a function

Key point 1: Numerically, many smooth functions are of low rank.

$$f(x, y) \approx \sum_{j=1}^{k} \sigma_j c_j(y) r_j(x).$$

Definition (Numerical low rank)

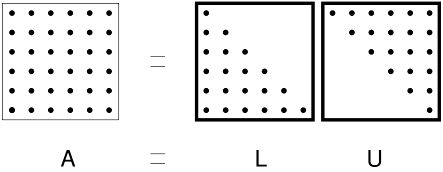
A function of numerical rank k and numerical degree (m, n) is of **low rank** if

$$k \cdot (m + n) < m \cdot n.$$

We're interested in approximating functions by low rank functions expressed in a Chebyshev basis.

Key point 2: Gaussian elimination can be used for low rank function approximation. [T. & Trefethen, 2013a]

The standard point of view:



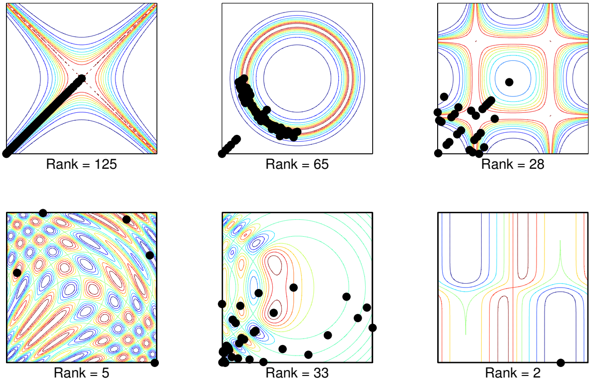
A different, equally simple point of view:

$$A \longleftarrow A - A(j,:)A(:,k)/A(j,k) \qquad \text{(GE step for matrices)}$$

$$f \longleftarrow f - f(x,:)f(:,y)/f(x,y) \qquad \text{(GE step for functions)}$$

Each step of GE is a rank-1 update.

```
f = chebfun2(@(x,y) cos(10*(x.^2+y))+sin(10*(x+y.^2)));
contour(f,'.')
```



2D Chebyshev technology: vector calculus, global rootfinding, quadrature, phase portraits, surfaces, etc.

Related work by mugshots

Eugene Tyrtyshnikov

Bebendorf, Goreinov, Oseledets, Savostyanov, Zamarashkin.

Mario Bebendorf

Gesenhues, Griebel, Hackbusch, Rjasanow.

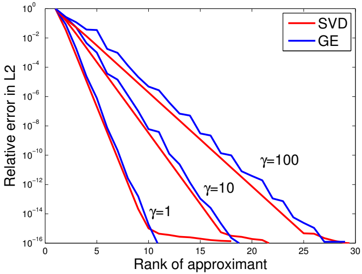
Keith Geddes

Carvajal, Chapman.

Key point 3: SVD is optimal, but Gaussian elimination can be better.

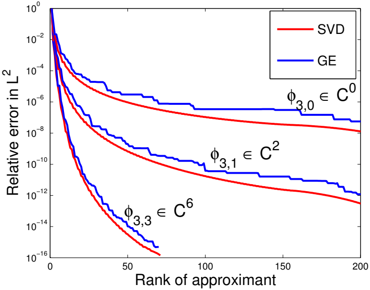
2D Runge function:

$$f(x,y) = \frac{1}{1+\gamma(x^2+y^2)}.$$



Wendland's CSRBFs:

$$f_k(x,y) = \phi_{3,k}(\|x-y\|_2) \in \mathcal{C}^{2k}.$$



Key point 4: 2D computations can exploit 1D technology. [T. & Trefethen, 2013b]

$$f(x,y) \approx \sum_{j=1}^{k} \sigma_j c_j(y) r_j(x).$$

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy \approx \sum_{j=1}^{k} \sigma_j \int_{-1}^{1} c_j(y)dy \int_{-1}^{1} r_j(x)dx.$$

Integration takes $\mathcal{O}(n \log n + kn)$ operations using 1D Clenshaw–Curtis quadrature.

```
F = @(x,y) exp(-(x.^2 + y.^2 + cos(4*x.*y)));

QUAD2D:   I = 1.399888131932670  time = 0.0717 secs
SUM2:     I = 1.399888131932670  time = 0.0097 secs
```

Definition (Tensor product operators)

$$\mathcal{L} = \mathcal{L}_y \otimes \mathcal{L}_x, \qquad \mathcal{L}_x \text{ and } \mathcal{L}_y \text{ linear.}$$

Example of tensor product operators

1. Differentiation

$$\mathcal{L} = \frac{\partial}{\partial y}, \qquad \mathcal{L} = \mathcal{D} \otimes \mathcal{I}, \qquad \mathcal{O}(kn) \text{ operations.}$$

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2. Evaluation

$$\mathcal{E}f = f(x,y), \qquad \mathcal{E} = \mathcal{E}_y \otimes \mathcal{E}_x, \qquad \mathcal{O}(kn^2) \text{ operations.}$$

Others include (Can you guess what they do?):
`sum2(f)`, `diff(f)`, `f(x,y)`, `sum(f)`, `cumsum2(f)`, `flipud(f)`, `trace(f)`, `gradient(f)`, `chebpoly2(f)`, `svd(f)`

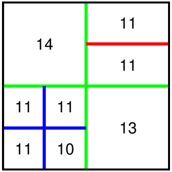
Key point 5: High degree global rootfinding can be done in 2D! [Nakatsukasa, Noferini, & T., 2013]

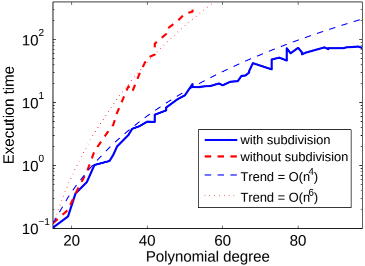
Problem: Find **all** (x, y) in $[-1, 1] \times [-1, 1]$ such that

$$\begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} = 0.$$

Uses **Chebyshev basis**, **2D subdivision**, and Bézout resultants.

sin((x–1/10)y)cos(1/(x + (y–9/10) + 5))
= (y–1/10)cos((x+(y+9/10)^2/4)) = 0

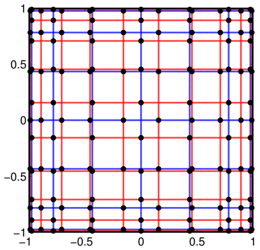




Find the solutions to:

$$\begin{pmatrix} T_7(x)T_7(y)\cos(xy) \\ T_{10}(x)T_{10}(y)\cos(x^2y) \end{pmatrix} = 0.$$

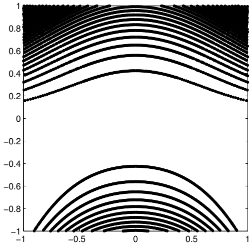
All the solutions line up along coordinate directions, but that's OK.



Find the solutions to:

$$\begin{pmatrix} \mathrm{Ai}(-13(x^2y + y^2)) \\ J_0(500x)y + xJ_1(500y) \end{pmatrix} = 0,$$

which has numerical degree of 580. There are 5932 solutions found in 501 seconds.



Global minimisation

Suppose $A \in \mathbb{R}^{n\times n}$ is a rank-k matrix,

$$A = \sum_{j=1}^{k} u_j v_j^T.$$

The minimum entry is the solution to the following problem:

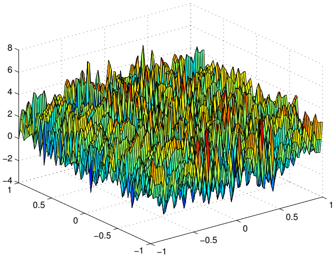
$$\min_{p,q} A_{pq} = \min_{p,q} \sum_{j=1}^{k} u_{pj} v_{qj} = \min_{p,q} \underline{u}_p^T \underline{v}_q.$$

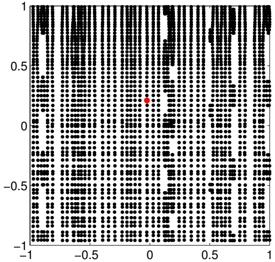
For $U = \{\underline{u}_1, \ldots, \underline{u}_n\} \in \mathbb{R}^{k\times n}$ and $V = \{\underline{v}_1, \ldots, \underline{v}_n\} \in \mathbb{R}^{k\times n}$:

$$\begin{aligned}\min_{p,q} \underline{u}_p^T \underline{v}_q &= \min\left\{\underline{u}^T \underline{v} : \underline{u} \in U, \underline{v} \in V\right\} \\ &= \min\left\{\underline{u}^T \underline{v} : \underline{u} \in \mathrm{conv}(U), \underline{v} \in \mathrm{conv}(V)\right\}.\end{aligned}$$

1. Find the minimum on a grid using convex hull.
2. Constrained Newton iteration.

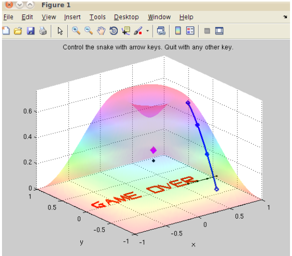
$$f(x,y) = \left(\frac{x^2}{4} + e^{\sin(50x)} + \sin(70\sin(x))\right) + \left(\frac{y^2}{4} + \sin(60e^y) + \sin(\sin(80y))\right) - \cos(10x)\sin(10y) - \sin(10x)\cos(10y).$$

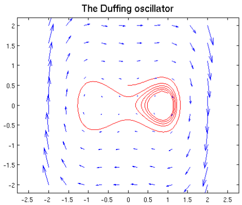


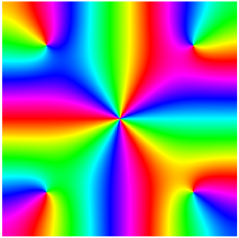


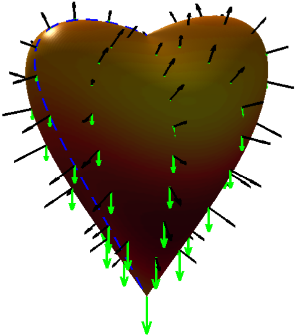
Chebfun2 calculates the global minimum in 0.72 seconds.

Lots of stuff we missed out









Future work and many questions

- More about Chebfun2's bivariate rootfinding. **Vanni Noferini's talk. You missed it.**
- More about Chebfun2 and surfaces. **See Rodrigo Platte's talk in San Diego.**
- What about higher dimensions? **Talk to Daniel Kressner.**
- What about PDEs on rectangles? **Talk to me in September.**
- What about on arbitrary domains? **Don't know. Help!**
- What else can Chebfun2 do? **Check out the website!**

References

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