

LIE THEORY AND TOPOLOGY

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1. MOTIVATION OF LIE GROUPS

Define a **topological Lie group** to be a group object in the category of topological manifolds. **Until stated otherwise G is assumed finite-dimensional.**

Theorem 1.1. *Let G be a compact, simple Hausdorff group. Then either G is a finite simple group or it is Lie.*

The latter are much easier to classify: they are $U(\mathfrak{n}), O(\mathfrak{n}), U(\mathfrak{n}, \mathbb{H})$ mod their centers, or one of five special cases (of dimensions ≤ 248), each more or less blamable on the octonions.

Nonexamples: profinite groups are compact and not Lie, but not simple.

Theorem 1.2. *Any closed subgroup of a Lie group is Lie.*

We'll define a **Lie group** to be a group object in the category of smooth manifolds. The principal example is $GL_n(\mathbb{R})$.

Theorem 1.3. *Any topological Lie group is uniquely (equivariantly) smoothable, and indeed, uniquely real-analytic.*

Moreover, any measurable homomorphism of Lie groups is automatically continuous smooth real-analytic.

However, not every Lie group is real algebraic, one standard example being $\widetilde{SL_2(\mathbb{R})}$. Proof: the center of an algebraic group is algebraic, and a discrete algebraic variety (of finite type) should be finite, but $Z(\widetilde{SL_2(\mathbb{R})}) \cong \pi_1(SL_2(\mathbb{R})) \cong \mathbb{Z}$.

Differential topology is a great place to work, in that any smooth Lie group comes with an **adjoint action** on the tangent space \mathfrak{g} at the identity. Differentiating that map $G \rightarrow \text{End}(\mathfrak{g})$ at the identity, we get the map $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$.

Theorem 1.4. *Each connected normal subgroup H of G gives a subrepresentation $\mathfrak{h} \leq \mathfrak{g}$.*

The group structure on G induces a "Lie algebra" structure on \mathfrak{g} , but we'll have very little use for it. In particular there are many subalgebras that don't come from *closed* subgroups, because of irrational-flow-on-a-torus problems. It is interesting to note, though, that every Lie algebra has a faithful matrix representation, and that the map ad only depends on the Lie algebra.

Hereafter the term **simple group** will mean one whose normal subgroups are *finite* rather than trivial.

Theorem 1.5. *If G is connected and simple, its normal subgroups are central, and $G/Z(G)$ is simple in the usual sense.*

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2. REDUCTION TO COMPACT GROUPS

Recall that any group has a unique maximal normal solvable subgroup. In a topological group we can add the adjective “connected”.

Define the **Killing form** on \mathfrak{g} to be

$$\langle X, Y \rangle := \text{Tr}(\text{ad } X \cdot \text{ad } Y).$$

This extends to a $G \times G$ -invariant pseudoRiemannian metric on G .

Theorem 2.1. (1) *If G is simple, this metric is unique.*

(2) *If G is compact, this metric is negative semidefinite, and its kernel is tangent to $Z(G)$.*

(3) *If G is simple and this metric is definite, then G is compact. (Since ad only depends on the Lie algebra, this says that if G, G' have the same Lie algebra and G is compact with finite center, then G' is also compact.)*

(4) *The radical of the Killing form is the tangent space to the unique maximal normal solvable connected subgroup.*

A group is **semisimple** if its Killing form is nondegenerate.

Theorem 2.2. (1) *(Levi 1905) There exists a semisimple “Levi subgroup” complementary to the radical of the Killing form.*

(2) *(Mal'cev 1942) Any two such choices are conjugate by elements of the maximal normal solvable connected subgroup.*

So far we've split the group into the kernel of \langle, \rangle and a semisimple complement. We can go further inside the semisimple part, essentially splitting into positive and negative parts:

Theorem 2.3. *(Iwasawa 1949) Let G be connected and semisimple. Let K be a maximal compact connected subgroup. Then there exists a complementary subgroup $A \times N$, where A is Ad-diagonalizable and N is Ad-unipotent, and AN is diffeomorphic to a vector space. In particular K is a deformation retract of G .*

There is another decomposition $G = KP$, where P is the exponential of $\mathfrak{k}^\perp \leq \mathfrak{g}$ and is not a subgroup.

If G is real algebraic, N will be too but A need not be. We can replace A by its Zariski closure A' , but then the map $K \times A' \times N \rightarrow G$ is only onto not bijective.

3. COMPACT CONNECTED LIE GROUPS

We'll usually use K for a compact connected Lie group.

Example 1. S^1 .

Example 2. $T = (S^1)^n$, called an **n-torus**.

Example 3. If $\pi_1(K)$ is finite, then \tilde{K} .

Theorem 3.1. *If $Z(K)$ is finite, then $\pi_1(K)$ is too.*

Example 4. $K_1 \times K_2 \times \dots \times K_m$.

Example 5. K/Γ , where $\Gamma \leq Z(K)$.

By the long exact sequence, if $\pi_1(K) = 0$, then $\pi_1(K/\Gamma) = \Gamma$.

Theorem 3.2. *Any K is a quotient of $K' := T \times K_1 \times \dots \times K_m$ by a finite central subgroup, where each K_i is simple and simply-connected. We can take $T = Z(K)_0$, and $K_1 \times \dots \times K_m$ as the universal cover of the commutator subgroup K' .*

Note that the quotient map $K' \rightarrow K$ induces an isomorphism of their Lie algebras.

Each K_i is real algebraic, so K is algebraic too. Indeed, there is a complex matrix group $K^{\mathbb{C}}$ containing K as a subgroup, homotopy retract, and fixed points of an antiholomorphic involution.

In particular, if K has a finite center, then there are finitely many connected groups with its Lie algebra, and they are all compact. (Whereas a torus has the same Lie algebra as a vector space.)

Example 6. $U(n) \cong (T^1 \times SU(n))/Z_n$.

Example 7. $SU(2)/Z_2 \cong SO(3)$. That gives three obvious quotients of $SU(2) \times SU(2)$. The nonobvious one turns out to be $SO(4)$, where one thinks of $U(1, \mathbb{H}) \times U(1, \mathbb{H})$ acting on \mathbb{H} by left and right multiplication. People make a big deal about this when studying the hydrogen atom, e.g. <http://math.ucr.edu/home/baez/classical/runge.pdf>.

Example 8. For $n > 2$, $\pi_1(SO(n)) = Z_2$. (Proof: $SO(3) \cong \mathbb{R}P^3$, and we get a long exact sequence on homotopy using the fibration over the sphere $SO(n)/SO(n-1)$.) Its universal cover is called $Spin(n)$. $Z(SO(\text{odd})) = 1$, so $Z(Spin(\text{odd})) \cong Z_2$, but $Z(SO(\text{even})) \cong Z_2$, so $|Z(Spin(\text{even}))| = 4$. Strangely, that center is Z_4 for $n \equiv 2 \pmod{4}$, $Z_2 \times Z_2$ for $n \equiv 0 \pmod{4}$.

Example 9. $Spin^c(n) := (Spin(n) \times S^1)/(Z_2)_\Delta$, so quotients to $SO(n)$ but not to $Spin(n)$.

Example 10. $SU(n)$, $SO(n)$, $U(n, \mathbb{H})$. These collide at $U(1, \mathbb{H}) \cong SU(2) \rightarrow SO(3)$, also at $Spin(4) \cong SU(2)^2$ as mentioned above, again at $U(2, \mathbb{H}) \cong Spin(5)$, and also at $Spin(6) \cong SU(4)$ (as the image of $SU(4) \rightarrow U(Alt^2 \mathbb{C}^4)$ is $SO(6)$),

Example 11. $G_2 = Aut(\mathbb{O})$ is 14-dimensional, simple, centerless, simply-connected.

Example 12. $F_4 = Aut(Herm_3(\mathbb{O}))$ is the automorphism group of the “exceptional Jordan algebra” of real dimension 27, the 3×3 octonionic-Hermitian matrices, under anticommutator. F_4 is 56-dimensional, centerless, simply-connected.

Example 13. E_6 also acts on the Jordan algebra, but only preserving the “determinant” 3-form, or one can see it as the collineation group of the octonionic projective plane. (Note that collineation groups are usually semidirect products, with one factor being the automorphism group of the skew-field involved!) It is 78-dimensional, and the center of its simply-connected form is Z_3 .

Example 14. E_7 is a 133-dimensional group, that acts on a 64-fold known as the “quaternionic projective plane”, and the center of its simply-connected form is Z_2 .

Example 15. E_8 is 248-dimensional, centerless, and simply-connected. Its smallest linear representation is its adjoint representation, which is pretty obnoxious if you ask me.

4. AVERAGING

Theorem 4.1. (Haar 1933) *On any locally compact group G there is a left-invariant measure, unique up to scale. Its total volume is finite exactly if G is compact (in which case we often scale it to be 1).*

If G has no one-dimensional representations, then its left-invariant measures are also right-invariant. (Non-example: the $\{ax + b\}$ solvable group.)

If G is Lie, existence is really easy; we even have left-invariant volume forms (unique up to scale).

Theorem 4.2. *On a compact group K , there exists a biinvariant Riemannian metric. The exponential map $\mathfrak{k} \rightarrow K$ defined using its geodesics is K -equivariant.*

(If K has discrete center, the negative of the Killing form will do.)

Proof. Pick any metric and average it under the $K \times K$ -action. □

Theorem 4.3. *If K acts on a real resp. complex vector space V , then V can be given a K -invariant orthogonal resp. Hermitian form.*

Thus, any K -subrepresentation of V has a K -invariant complement.

Thus, any finite-dimensional representation of K is a direct sum of irreducible representations.

Lots more to say about that, of course! Let's start with

Lemma 4.4. *Let T act linearly on $V = \mathbb{R}^n$. Then V is the direct sum of V^T and a bunch of 2-dimensional irreps. In particular $\dim_{\mathbb{R}} V \equiv \dim_{\mathbb{R}} V^T \pmod{2}$.*

If T acts linearly on $V = \mathbb{C}^n$, then V is a direct sum of complex-one-dimensional irreps.

Proof. Pick a T -invariant orthogonal form on V , so we can split complements U^\perp to subrepresentations.

If $\vec{v} \in V \otimes \mathbb{C}$ is an eigenvector for t , then $\vec{v} + \bar{\vec{v}}, i(\vec{v} - \bar{\vec{v}})$ generate a T -subrep $U \cong \mathbb{R}^2$. If we've already split off the invariants, then U must be irreducible, since any homomorphism of $T \rightarrow O(1)$ is trivial.

The complex case is similar but simpler. □

5. MAXIMAL TORI AND WEYL GROUPS

A **maximal torus** T of a compact Lie group is what you'd expect.¹ Let $t \in T$ be a topological generator, i.e. $\overline{\langle t \rangle} = T$ (that's most elements). Then

$$K \cdot t \cong K/C_K(t) \cong K/C_K(T)$$

and since T is a *maximal* torus, the group $C_K(T)/T$ must be finite, so this space $K \cdot t$ is a finite quotient of K/T . (Later we'll see that if K is connected, then $C_K(T) = T$.)

The fixed points of t on K/T are also fixed by T ;

$$TkT/T = kT/T \iff k \in N(T)$$

so the fixed points correspond to $W := N(T)/T$, the **Weyl group**.

Lemma 5.1. (1) *If K is compact positive-dimensional, then K contains a circle.*

(2) *If T is maximal, then $C_K(T)/T$ is finite. (Later we will show that if K is connected, $C_K(T) = T$.)*

¹For noncompact groups, one generalizes the notion of "torus" to a connected group that acts diagonalizably on \mathfrak{g} , so e.g. the diagonal matrices in $SL_n(\mathbb{R})$ count as a "torus".

Proof. Pick $X \in \mathfrak{k} \setminus 0$, and let $J = \overline{\exp(\mathbb{R} \cdot X)}$. Then J is connected, and abelian, so $\exp : \mathfrak{j} \rightarrow J$ is a group surjection. Since J is connected and Hausdorff, $\ker \exp$ is a lattice so J is a torus. Then tori contain circles.

If $C_K(T)/T$ isn't finite, then we can choose a circle subgroup J' in $C_K(T)$, and take its preimage J in $C_K(T)$. This is a compact connected group, so we know what it looks like. But for T to be codimension one in it, J has to be a torus. In which case T wasn't maximal. \square

Theorem 5.2. *The Weyl group is finite.*

Proof. We need to show $N_K(T)_0 = T$, so W is discrete, but also compact hence finite. We already noted that $C_K(T)_0 = T$.

The kernel of the conjugation action of $N(T)$ on T is $C_K(T)$, so there is a faithful action of a finite quotient of W on T . But T 's automorphism group is $GL_{\dim T}(\mathbb{Z})$, so discrete. \square

If we can pick $X \in \mathfrak{t}$, such that $\exp(X) = t$, then the isolated fixed points says that X 's vector field has isolated zeros. To apply Poincaré-Hopf, we need to compute indices:

Lemma 5.3. *If $V^T = \{\vec{0}\}$, and $X \in \mathfrak{t}$ such that $t := \exp(X)$ topologically generates T , then the index of X 's unique fixed point is 1.*

Proof. By assumption, $V = \oplus U_i$, where U_i is a 2-d irrep of T . The homomorphism $T \rightarrow O(U_i) \cong O(2)$ lands inside $SO(2)$, so gives index 1. When we take the sum of all the irreps, we take the product of the indices and get 1. \square

Theorem 5.4. (1) *The Euler characteristic of K/T is $|W| \neq 0$.*

(2) *The Lefschetz number of any k acting on K/T is $|W| \neq 0$.*

(3) *Any element is contained in a conjugate of T .*

(4) *Any two maximal tori are conjugate, in particular, all of the same dimension, called the rank of K . (E.g. "Unitary matrices are diagonalizable.")*

Proof. (1) The vector field given by X is almost complex, so all the indices are positive. (Actually they're 1.)

(2) Since K is connected, any k 's Lefschetz number is the Euler characteristic.

(3) Since k must have a fixed point, $kxT = xT$, we learn $k \in xTx^{-1}$.

(4) Let k be a topological generator of a second maximal torus, and apply the previous. \square

One view is that maximal tori are analogous to Sylow subgroups, and this $\chi \neq 0$ argument takes the place of the counting argument that one uses to prove that all p -Sylows are conjugate. We'll see another application of this analogy later.

Factoid: if K is simply connected, then every abelian subgroup lies in a maximal torus.

Non-examples:

- The diagonal matrices in $SO(3)$ don't live in any maximal torus ($SO(2)$, or a conjugate).
- There exist infinite-dimensional symplectomorphism groups with finite-dimensional maximal tori of differing dimensions.

The idea that we can cover a group using conjugates of an abelian subgroup will turn out to be incredibly great. Note that we can *never* cover a finite group G with conjugates of a proper subgroup H (even nonabelian):

$$|\cup_{g \in G} g \cdot H| = |\cup_{g \in G/N(H)} g \cdot H| < |\prod_{g \in G/N(H)} g \cdot H| = \sum_{g \in G/N(H)} |g \cdot H| = |G/N(H)| |H| = |G|/|N(H)/H| \leq |G|.$$

The first \leq can only be $=$ if $N(H) = G$, and the second only if $N(H) = H$.)

6. MORSE THEORY ON COADJOINT ORBITS

It'd be nice if X 's vector field was something like the gradient of a Morse function, so we could do Morse theory on $K \cdot \mathfrak{t}$. The natural place for $X \in \mathfrak{k}$ to induce a function is on \mathfrak{k}^* . Since exponential maps are local diffeomorphisms, we get to roughly correspond the conjugation orbits of K on K with the orbits of K on \mathfrak{k} , and on \mathfrak{k}^* (again using the K -invariant metric).

One reason that \mathfrak{g}^* is a great space to work on, in general, is that it has an interesting Poisson structure, which is a shadow of the fact that

$$\text{Fun}(\mathfrak{g}^*) = \text{Sym}(\mathfrak{g}) = \text{gr } U\mathfrak{g}.$$

The Poisson bivector $\pi \in \Gamma(\text{Alt}^2 T\mathfrak{g}^*)$ is easy to write down:

$$\pi(X, Y)|_\lambda := \lambda([X, Y]), \quad X, Y \in (T_\lambda \mathfrak{g}^*)^* \cong \mathfrak{g}$$

Its symplectic leaves turn out to be exactly the orbits of G , called **coadjoint orbits**.

Example: $K = U(n)$. Then we can equivariantly identify \mathfrak{k}^* with the space of Hermitian matrices, and the orbits \mathcal{O}_λ are isospectral sets, i.e. correspond to spectra $(\lambda_1 \geq \dots \geq \lambda_n)$. The simplest case is $\lambda = (1, 1, 1, \dots, 1, 0, \dots, 0)$ with k 1s, in which case the map $M \mapsto \text{Image}(M)$ corresponds \mathcal{O}_λ with the Grassmannian of k -planes.

More generally, we can take M to the nested list (\sum top i eigenspaces), corresponding it to a partial flag manifold $\{(V_1 < V_2 < \dots < \mathbb{C}^n)\}$. It is rather amazing that the *real* vector space \mathfrak{k}^* has been naturally partitioned into compact *complex* manifolds! (Of course symplectic manifolds are always even-real-dimensional, but they are not always complex [Thurston '76].)

In Morse theory, one uses a Riemannian metric to build a vector field from a function (or really, from the 1-form obtained as its derivative). What is perhaps unsatisfying is that the resulting vector fields do not annihilate the metric or the function. If one uses a Poisson bivector instead, the resulting "Hamiltonian vector fields" *do* annihilate the Poisson tensor and the function.

Theorem 6.1. *If K is compact, then any coadjoint orbit can be given a K -invariant Riemannian metric compatible with its symplectic structure, such that the result is almost Kähler.*

(In fact it will be honestly complex, but we won't use that.)

Proof. Pick a compatible almost complex structure, get a metric, average it. \square

Lemma 6.2. *Let T act on a Hermitian vector space $V \cong \oplus_{\lambda_i} \mathbb{C}_{\lambda_i}$, where each \mathbb{C}_{λ_i} is an irrep with character $\lambda \in T^*$. Then for $X \in \mathfrak{t}$, the function*

$$f_X : V \rightarrow \mathbb{R}, \quad (z_1, \dots, z_m) \mapsto \sum_i -\frac{1}{2} |z_i|^2 \langle X, \lambda_i \rangle$$

has Poisson gradient equal to the action of X , and is unique with this property up to addition of a constant.

If no $\langle X, \lambda_i \rangle = 0$, then it is a Morse function, with index $2\#\{\lambda_i : \langle X, \lambda_i \rangle < 0\}$.

Theorem 6.3. K/T has an even-dimensional cell decomposition, with cells indexed by W . In particular, $\chi(K/T) = |W|$, and K/T is simply-connected. Moreover $C_K(T) = T$.

Proof. Fix a K -invariant metric on \mathfrak{k} . Pick $\lambda \in \mathfrak{k}^*$ such that the corresponding $\lambda^* \in \mathfrak{k}$ is within the injectivity radius of $\exp : \mathfrak{k} \rightarrow K$, and such that $t := \exp(\lambda^*)$ topologically generates T . Then

$$K \cdot \lambda^* \cong K \cdot t \cong K/C_K(t) \cong K/C_K(T).$$

On \mathfrak{k}^* , we have the linear functional $X \cdot$, which restricted to $K \cdot \lambda^*$ gives a function. We can analyze its critical points using the Hamiltonian vector field (which is *not* the Riemannian gradient) to determine that they are isolated and even index. \square

Let T be a maximal torus, $t \in T$ a topological generator within the injectivity radius of \exp , and $X \in \mathfrak{t}$ its logarithm. Then X defines a map

$$\Phi_X : K \cdot X \rightarrow \mathbb{R}, \quad \mu \mapsto \langle \mu, X \rangle$$

which will turn out to give a Morse decomposition into even-dimensional cells, generalizing the usual one for projective space. In the next section we figure out what the dimensions are of those cells.

6.1. The Weyl group is a reflection group. Note first that W acts on \mathfrak{t} preserving the Killing form, so, orthogonally.

Theorem 6.4. Let T act on $\mathfrak{g}/\mathfrak{t}$, a sum of 2-irreps.

- (1) These irreps U are all nonisomorphic.
- (2) Each $\mathfrak{t} + u$ is the tangent space to a subgroup L_U isomorphic to $SU(2) \times T^{n-1}$, possibly mod $(Z_2)_\Delta$.
- (3) $W(L_U) \leq W$ is isomorphic to Z_2 .

Because complex representation theory is easier than real, we'll often complexify \mathfrak{k} to $\mathfrak{k}_\mathbb{C} := \mathbb{C} \otimes \mathfrak{k}$. We'll use T^* to denote the set of one-dimensional complex representations or **weights** of T , which we can identify with a lattice in \mathfrak{t}^* by

$$\lambda : T \rightarrow U(1) \quad \mapsto \quad \lambda' : \mathfrak{t} \rightarrow i\mathbb{R} \quad \mapsto \quad i\lambda' \in \mathfrak{t}^*.$$

Then W preserves T^* inside \mathfrak{t}^* .

Example: $K = U(n)$, $W \cong S_n$, $T^* \cong \mathbb{Z}^n$, with the usual action.

The **root system** $\Delta \subset T^*$ of K is the set of weights in $(\mathfrak{g}/\mathfrak{t}) \otimes \mathbb{C}$. So each U in the above decomposition gives a pair $\pm\beta$ of such roots, and we'll use L_β or $L_{-\beta}$ in place of L_U .

If we pick a random functional $X \in \mathfrak{t}$, i.e. no $\langle X, \beta \rangle = 0$, then we can use it to define a **positive system** $\Delta_+ \subset \Delta$. Let $\Delta_1 \subseteq \Delta_+$ denote the **simple roots**, that aren't sums of other positive roots.

Theorem 6.5. (1) The action of $W(L_U)$ on T^* is by reflections.

- (2) Δ_1 is linearly independent.
- (3) The reflections $\{r_\alpha \in W(L_\alpha)\}_{\alpha \in \Delta_1}$ generate W .

(4) Δ_1 spans a lattice inside T^* of index $|Z(K)|$, called the **root lattice**.

Proof. (1) We already calculated that the action of $W_{U(n)}$ on T^* is the standard action of S_n on \mathbb{Z}^n , which barely modified tells us that $W_{SU(2)}$ acts on its $T^* \cong \mathbb{Z}$ by reflection (negation). Enlarging $SU(2)$ to $SU(2) \times T^{n-1}$ just gives the reflection a large invariant space.

Because K preserves the definite form on \mathfrak{t}^* , $N(T)$ preserves the restriction of that form to \mathfrak{t}^* .

- (2) The proof goes by a better fact: each $\langle \alpha, \alpha' \rangle \leq 0$. Anyway it's findable in any Lie textbook.
- (3) This is too. It's really a statement about reflection groups, not Lie theory. (Note, though, that it uses K connected; otherwise we could take the Z_3 half of $N(T)$ inside $SU(3)$.)
- (4) The kernel in K of the adjoint action is $Z(K) \leq C_K(T) = T$; write $\text{Ad } T$ for $T/Z(K)$, a maximal torus of $K/Z(K)$. The $U(1)$ -dual of the exact sequence of compact groups

$$1 \rightarrow Z(K) \rightarrow T \rightarrow \text{Ad } T \rightarrow 1$$

is the exact sequence of discrete groups

$$0 \longleftarrow Z(K)^* \longleftarrow T^* \longleftarrow \text{root lattice} \longleftarrow 0.$$

http://en.wikipedia.org/wiki/Pontryagin_duality

□

At this point we can jack into the purely combinatorial theory of reflection groups, to learn things like

Theorem 6.6. *If $w \in W$, then $|\Delta_+ \setminus w \cdot \Delta_+| = \ell(w)$, the **length** of w written as a minimal product of simple reflections.*

Corollary 6.7. *The Morse cell of Φ_x , coming down from the point $wT/T \in K/T$, has dimension $2\ell(w)$. In particular, if $Z(K)$ is finite then $H^2(K/T) \cong \mathbb{Z}^{\dim T}$.*

Define the **height** of a root $\beta \in \Delta$ as the sum of the coefficients when β is expanded in the simple roots Δ_1 . (Then if we let Δ_k denote the set of roots of height k , it motivates the notation Δ_{\pm} and Δ_1 .)

7. TOPOLOGY OF K/T

The long exact sequence on homotopy for $T \rightarrow K \rightarrow K/T$ gives us

$$\dots \rightarrow \pi_2(T) \rightarrow \pi_2(K) \rightarrow \pi_2(K/T) \rightarrow \pi_1(T) \rightarrow \pi_1(K) \rightarrow \pi_1(K/T) \rightarrow \pi_0(T) = 1.$$

We know the homotopy groups of T . But let's be more precise than just to say $\pi_1(T) \cong \mathbb{Z}^{\dim T}$; we can identify it as a group with the **coweight lattice** Λ , the kernel of the exponential map $\mathfrak{t} \rightarrow T$. (This is the \mathbb{Z} -dual of the weight lattice T^* .)

$$1 \rightarrow \pi_2(K) \rightarrow \pi_2(K/T) \rightarrow \Lambda \rightarrow \pi_1(K) \rightarrow \pi_1(K/T) \rightarrow 1$$

Since K/T is simply-connected, we can kill the last one, *and* invoke Hurewicz:

$$1 \rightarrow \pi_2(K) \rightarrow H_2(K/T) \rightarrow \Lambda \rightarrow \pi_1(K) \rightarrow 1$$

So in particular $\pi_1(K)$ is abelian (but the Eckman-Hilton argument lets one see that for any group).

The \mathbb{Z} -dual of this map $H_2(K/T) \rightarrow \Lambda$ is a map $H^2(K/T) \leftarrow T^*$, and turns out to be the map $\lambda \mapsto c_1(K \times^T \mathbb{C}_\lambda)$ taking a representation to the first Chern class of its “associated bundle”, under the associated-bundle construction for the principal bundle $K \rightarrow K/T$.

Theorem 7.1. $\pi_2(G) = 1$ for any finite-dimensional Lie group.

Proof. Retract G to K . The above sequence says $\pi_2(K)$ is free abelian, so nonzero iff $H^2(K; \mathbb{R}) \neq 0$. We can compute the latter using de Rham cohomology, as follows. Inside the complex of forms lies the finite-dimensional subcomplex of left-invariant forms, and we can average forms to show that the inclusion induces a homotopy equivalence of the complexes. Then the “Lie algebra cohomology” $H^2(\mathfrak{k})$ turns out to be measuring nontrivial central extensions, and there are none. \square

So for instance, any principal bundle over S^3 with finite-dimensional structure group must be trivial! (Of course the tangent bundle is, because S^3 is a group, but not every Z_2 -bundle over S^1 is trivial, for instance.)

Corollary 7.2. If K is centerless, then $\pi_1(K)$ is finite.

Proof. If K is centerless, then Δ_1 spans T^* , so in

$$1 \rightarrow H_2(K/T) \rightarrow \Lambda \rightarrow \pi_1(K) \rightarrow 1$$

$H_2(K/T)$ and Λ have the same rank. \square

It turns out that $\pi_3(K)$ is always \mathbb{Z} for a compact simple group (e.g. $SO(5)$ but not $SO(4)$). Basically, each $L'_\beta \rightarrow K$ gives the generator of π_3 , remembering that $\widetilde{L'_\beta} \cong S^3$.

7.1. Schubert calculus. Let $S_w \in H^*(K/T)$ denote the cohomology class dual to the cycle X_w flowing down into the point w , so $\deg S_w = 2\ell(w)$, and $c_{uv}^w \in \mathbb{Z}$ denote the structure constants in the multiplication $S_u S_v = \sum_w S_w$.

Theorem 7.3. (Kleiman 1973) Each c_{uv}^w is nonnegative.

Proof. In fact K/T is a complex manifold, and each X_w is a subvariety, called a **Schubert variety**.

We can compute $\int_{K/T} S_w S_v$ by intersecting X_v with $w_0 \cdot X_w$. Morse-theoretically, we get the closure of the union of the lines from $w_0 w$ down to v .

- (1) If $w_0 w \not\geq v$, this is empty.
- (2) If $w_0 w > v$, this is positive-dimensional.
- (3) If $w_0 w = v$, this is just a point.

So $\int_{K/T} S_w S_v = \delta_{w_0 w, v}$. (N.B. We need to use the fact that this Morse function is Palais-Smale.)

Hence $c_{uv}^w = \int S_u S_v S_{w_0 w} = |X_u \cap (g \cdot X_v) \cap (h \cdot S_{w_0 w})|$ for generic $g, h \in K$, because the complex structure guarantees that each point in that triple intersection contributes 1 to the cohomological intersection. \square

There are many extensions of this result to other cohomology theories (e.g. equivariant quantum K-theory), but manifestly positive *formulæ* for very few of them. The simplest (and most important) case is when $K \cdot t$ is a Grassmannian, and there we do have many rules.

<http://www.math.cornell.edu/~allenk/plenary.pdf>

7.2. A non-topological applications of the Morse theory: Horn's inequalities.

Lemma 7.4. *Let f be a Morse function on M with C the set of critical points, and $M = \coprod_C M_c$ the Morse decomposition. If $m \in M_c$, then $f(m) \geq f(c)$.*

Theorem 7.5. (Helmke-Rosenthal 1995) *Let $H_a + H_b + H_c = 0$, where each H_d is a Hermitian matrix with spectrum $(d_1 \geq d_2 \geq \dots \geq d_n)$. Let (λ, μ, ν) be a triple of Schubert classes on $\text{Gr}_k(\mathbb{C}^n)$ such that $\int S_\lambda S_\mu S_\nu \neq 0$. Then*

$$\lambda \cdot a + \mu \cdot b + \nu \cdot c \leq 0$$

where we consider λ, μ, ν as vectors from $\{0^{n-k}1^k\}$.

Proof. Each H_d gives a Morse function $V \mapsto \text{Tr}(H_d \pi_V)$ on the k -Grassmannian, whose critical points come when V is a sum of eigenlines of H_d . By the integral, there exists a $V \in \text{Gr}_k(\mathbb{C}^n)$ in the intersection of the three Morse strata for the three different Morse-Schubert stratifications. Hence

$$0 = \text{Tr}(0) = \text{Tr}((H_a + H_b + H_c)\pi_V) = \text{Tr}(H_a \pi_V) + \text{Tr}(H_b \pi_V) + \text{Tr}(H_c \pi_V) \geq \lambda \cdot a + \mu \cdot b + \nu \cdot c.$$

□

Klyachko proved that these give *all* the inequalities on a, b, c . Belkale proved that it's enough to consider $\int S_\lambda S_\mu S_\nu = 1$. Tao, Woodward, and I proved that all those remaining inequalities are indeed necessary. Much more about this is at the URL above.

8. A HINT OF REPRESENTATION THEORY

Theorem 8.1. *Let V, W be reps of an arbitrary group G .*

- (1) $\text{Hom}(V, W)$ and $V^* \otimes W$ are reps, and the natural map $V^* \otimes W \rightarrow \text{Hom}(V, W)$ is G -equivariant.
- (2) If V is finite-dimensional, then $g \mapsto \text{Tr}(g|_V)$ is constant on conjugacy classes, and called the **character** of V .
- (3) If V, W are isomorphic finite-dimensional reps, they have the same character.
- (4) Let $\text{Hom}_G(V, W) := \text{Hom}(V, W)^G$, the equivariant maps or **intertwiners**. If V, W are irreducible, then $\dim \text{Hom}_G(V, W) = [V \cong W]$ (Schur's lemma).

If G is compact:

- (1) $\text{Tr}(g|_V) = \overline{\text{Tr}(g|_{V^*})}$.
- (2) Let $\pi_G|_V = \int_G g|_V$. Then π_G is a projection $V \rightarrow V^G$.
- (3) Consequently, if V is finite-dimensional, $\text{Tr}(g|_V) = \dim V^G$.

Theorem 8.2. *The characters of the irreps of a compact group K are orthonormal elements of the Hermitian vector space $L^2(K; \mathbb{C})$.*

Proof.

$$\begin{aligned} \langle \text{Tr}(g|_V), \text{Tr}(g|_W) \rangle &:= \int_K \overline{\text{Tr}(g|_V)} \text{Tr}(g|_W) = \int_K \text{Tr}(g|_{V^*}) \text{Tr}(g|_W) \\ &= \int_K \text{Tr}(g|_{V^* \otimes W}) = \int_K \text{Tr}(g|_{\text{Hom}(V, W)}) = \dim \text{Hom}(V, W)^K = \dim \text{Hom}_K(V, W) \end{aligned}$$

and that is 1 or 0 by Schur's lemma. \square

Of course, they're not a basis for $L^2(K)$, since they're constant on conjugacy classes; really we might hope that they be a basis for $L^2(K/\sim)$ (as indeed they are). We study that space in the next section.

Corollary 8.3. *Let V, W be reps of a compact connected group K , with maximal torus T . Then V, W are isomorphic if they are isomorphic as T -representations.*

The nicest way to write down a T -representation is as a function $T^* \rightarrow \mathbb{N}$, taking

$$\lambda \mapsto \dim \text{Hom}_T(\mathbb{C}_\lambda, V) = \int_T t^{-\lambda} \text{Tr}(t|_V).$$

If the representation comes from K , then this **multiplicity diagram** will be W -invariant. When $\dim \text{Hom}_T(\mathbb{C}_\lambda, V) > 0$, call λ a **weight of V** .

The biggest theorem in the subject requires a concept early from the next section.

9. CONJUGACY CLASSES

Each $\beta \in \Delta$ defines a hyperplane in \mathfrak{t} , and this gives a decomposition of \mathfrak{t} into top-dimensional cones called **Weyl chambers**. It is a wonderful result in finite reflection groups that W acts simply transitively on the set of Weyl chambers. Having picked Δ_+ , we have exactly broken this W -symmetry to have a **positive Weyl chamber**

$$\mathfrak{t}_+ := \{X : \langle X, \alpha \rangle \geq 0 \quad \forall \alpha \in \Delta_+\}$$

which provides a system of representatives:

$$\mathfrak{t}_+ \hookrightarrow \mathfrak{t} \twoheadrightarrow \mathfrak{t}/W.$$

This is one of the great benefits of W being a reflection group. (We will see an example later of how much more annoying moduli spaces are when quotienting by a group that isn't one.)

Because the simple roots are linearly independent, this cone is always an orthant times a vector space (whose dimension is that of $Z(K)$).

Recall that given $X \in \mathfrak{t}$ not perpendicular to any $\beta \in \Delta$, i.e. in the interior of some Weyl chamber, we can define $\Delta_+ := \{\beta \in \Delta : \langle X, \beta \rangle > 0\}$. Then defining \mathfrak{t}_+ as above, we obtain the chamber that contains X .

We can now state the big theorem in the subject of representations of compact connected groups, which uses the corresponding chamber in \mathfrak{t}^* .

Theorem 9.1 (of the highest weight). *Fix $X \in \mathfrak{t}$, defining a positive Weyl chamber \mathfrak{t}_+^* of a connected compact Lie group K . Then the map*

$$[V] \mapsto \arg \max_{\lambda \in T^*} \{\langle X, \lambda \rangle : \lambda \text{ is a weight of } V\}$$

taking an isomorphism class $[V]$ of K -irreps to its **highest weight** is a bijection $\{[\text{irreps}]\} \rightarrow T_+^*$.

Moreover, that $\dim \operatorname{Hom}_{\mathbb{T}}(\mathbb{C}_\lambda, V)$ is 1, i.e. the **high weight vectors** are unique up to scale.

How to think about the lower-dimensional walls in the Weyl hyperplane arrangement? If $\langle X, \beta \rangle = 0$, then $C_K(X) \geq L_\beta$, and vice versa. So as we go to smaller faces, the centralizer jumps dimension, and the K -orbit shrinks.

9.1. Conjugacy classes in \mathfrak{k}^* (coadjoint orbits). So far we've shown that every conjugacy class of K meets T , so using the exponential map, every orbit in \mathfrak{k} goes through \mathfrak{t} . (That only works within the injectivity radius, but we can rescale to work in there.) And then, we can use W to cut \mathfrak{t} down further to its positive Weyl chamber.

Using an invariant form, we can regard \mathfrak{t}^* as a *subspace* of \mathfrak{k}^* , and define a positive Weyl chamber there as well.

In both cases, though, we haven't discussed the issue of whether the chamber represents some K -conjugacy class more than once (it will turn out it does not).

Example: if $K = \operatorname{U}(n)$, and we naturally identify \mathfrak{t}^* with \mathbb{R}^n , then the usual positive Weyl chamber is $(\lambda_1 \geq \lambda_2 \geq \dots \lambda_n)$. If we restrict to $\operatorname{SU}(n)$, then it gets the additional condition $\sum \lambda_i = 0$, and becomes a pointed cone.

9.2. Conjugacy classes in K . So far T/W maps onto the space K/\sim of conjugacy classes. To know it's an isomorphism, we need

Lemma 9.2. *Two elements of T are K -conjugate iff they're $N(T)$ -conjugate.*

Proof. Let $t, gtg^{-1} \in T$ be the two elements, so $t \in g^{-1}Tg$. Let $H = C_K(t)_0$. Then $H \geq T, g^{-1}Tg$. Hence some $h \in H$ has $h^{-1}Th = g^{-1}Tg$, since all tori in the compact connected Lie group H are conjugate. So $hg^{-1} \in N(T)$, and

$$gtg^{-1} = gh^{-1}thg^{-1} = (hg^{-1})^{-1}t(hg^{-1}). \quad \square$$

The corresponding statement for Sylow subgroups of a finite group is called "Burnside's fusion theorem", and has very much the same proof.

Corollary 9.3. *The subset $\mathfrak{t}_+^* \subseteq \mathfrak{k}^*$ meets each coadjoint orbit exactly once.*

Call an element of K **regular** if it lies in a unique maximal torus, which includes the case of topological generators but is more general. Example: a unitary matrix is regular iff it has distinct eigenvalues (rather than their logs being incommensurable).

This property is obviously invariant under conjugacy, so it suffices to understand which elements of T are regular. If $t \in T$ lies in another torus, then $\dim C_G(t) > \dim T$, so t acts trivially on some part of $\mathfrak{k}/\mathfrak{t}$, i.e. $\beta(t) = 1$ for some $\beta \in \Delta$.

This is easiest to analyze up on \mathfrak{t} , where the condition is $\langle \beta, X \rangle \in \mathbb{Z}$, giving a decomposition of \mathfrak{t} into **Weyl alcoves**. Note that nearby 0, this is just the decomposition into Weyl chambers.

$$T/W \cong (\mathfrak{t}/\Lambda)/W \cong \mathfrak{t}/(W \ltimes \Lambda)$$

Theorem 9.4. *If K is simple and simply connected, then $W \ltimes \Lambda$ is an affine reflection group, whose new generator is reflection through the hyperplane $\langle \cdot, \alpha_0 \rangle \leq 1$ where α_0 is the **lowest root** in Δ , the one of greatest negative height.*

In particular, T/W can be identified with a simplex T_+ inside \mathfrak{t} .

- If \mathfrak{t} is on a face of dimension m of the positive Weyl alcove T_+ , the group $C_K(\mathfrak{t})$ has central rank m .
- So \mathfrak{t} is on a vertex iff $C_K(\mathfrak{t})$ is semisimple. The $\text{rank}(G) + 1$ many such conjugacy classes are called **special**.
- $Z(K)$ acts on T_+ by rigid motions, taking the identity vertex to the other “central vertices”.
- Hence $|Z(K)| \leq \text{rank}(K) + 1$. Equality holds exactly for $K = \text{SU}(n)$.
- To compute the order (resp. adjoint order) of a conjugacy class, scale it until it lies on a reflection of the identity vertex (resp. a central vertex).
- $W \cdot T_+$, considered inside \mathfrak{t} , is a polytope that tessellates \mathfrak{t} . The map $\exp : W \cdot T_+ \rightarrow T$ is onto, and one-to-one away from the boundary.
- If K has finite center, we can still use this technology to analyze K 's conjugacy classes by $K/\sim = (\tilde{K}/\sim)/\pi_1(K)$. But this may not be a polytope, as in $\text{PU}(3)$. (Or it may be, as in $\text{SO}(5)$.)

10. LOOP GROUPS

10.1. **Loop spaces.** Let M be a Riemannian manifold, and $LM = \text{Map}(S^1, M)$ be the space of smooth based loops into M . This is an infinite-dimensional Fréchet manifold, with tangent spaces

$$T_\gamma LM \cong \Gamma(S^1; \gamma^* TM), \quad \gamma \in LM.$$

We can define a metric on the loop space:

$$\langle \vec{v}, \vec{w} \rangle := \int_{S^1} \langle \vec{v}|_{\mathfrak{t}}, \vec{w}|_{\mathfrak{t}} \rangle$$

where the latter \langle, \rangle occurs inside $T_M \gamma(\mathfrak{t})$. We also can define a 1-form

$$\alpha(\vec{v}) = \int_{S^1} \langle \vec{v}|_{\mathfrak{t}}, \gamma'(\mathfrak{t}) \rangle$$

and take d of it to get a closed 2-form.

Finally, we can define an **action functional**

$$A(\gamma) = \int_{S^1} \frac{1}{2} |\gamma'(\mathfrak{t})|^2$$

whose critical points are the geodesic loops.

Theorem 10.1. *Where ω is nondegenerate, the “rotate the loop” vector field is the Hamiltonian vector field of the action functional.*

For a generic metric, the closed geodesics are isolated, and A is a Morse function – indeed, Morse invented Morse theory for this application.

One can do some amazing, if nonrigorous, stuff with this 2-form [At83]. But it's in some sense boring since it's d of a 1-form.

If M is a group G , then $TM \cong G \times \mathfrak{g}$, so each tangent space is isomorphic to $L\mathfrak{g}$. In particular, we can talk about the derivative of a tangent vector, and get another tangent vector!

10.2. The based loop group. Fix a compact connected Lie group K . (So the metric will be very *nongeneric*.)

The correct space to work on will be not LK , but $\Omega K = \text{Map}(S^1, K)$, the space of smooth based loops into K . Both of these are groups, under pointwise multiplication, something like a limit of K^n as the n points become dense in S^1 , and they are related by

$$LK/K \cong \Omega K, \quad \text{identifying } K \cong \{\text{constant loops}\}.$$

One benefit of this identification is to put a circle action on ΩK , which doesn't exist for general M . But each turns out to be the wrong group!

ΩK has very nice geodesics:

Theorem 10.2. $\gamma : S^1 \rightarrow K$ is a basepoint-preserving geodesic iff it is a one-parameter subgroup. We can conjugate it to lie in T , and then to get its generator to lie in the positive Weyl chamber \mathfrak{t}_+ .

So the space of critical points of A is a disjoint union of adjoint K -orbits, one for each dominant coweight in \mathfrak{t}_+ . When the dominant coweight lies in the interior, the orbit is a K/T .

The index of a stratum is the **height** of the coweight, and in particular, finite.

This is the example for which Bott invented Morse-Bott theory; A turns out to be a Morse-Bott function.

Also, there is a natural symplectic 2-form on ΩK :

$$\omega(\vec{v}, \vec{w}) = \int_{S^1} \langle \vec{v}', \vec{w} \rangle$$

which is antisymmetric by integration-by-parts. (It is even the imaginary part of a Kähler form on ΩK [Pr82].) Then the circle action is Hamiltonian, and generated by the energy functional.

Theorem 10.3. If G acts transitively and symplectically on a symplectic manifold M , then M is a cover of a central extension of G , but not necessarily of G itself.

(Even better: if M 's 2-form ω is the curvature of a Hermitian line bundle \mathcal{L} , then some central extension of G can have its action lifted to \mathcal{L} .)

Proof. On the Lie algebra level, $\mathfrak{g} \rightarrow \mathfrak{symp}(M)$. There is an exact sequence $0 \rightarrow H^0(M) \rightarrow C^\infty(M) \rightarrow \mathfrak{symp}(M) \rightarrow H^1(M) \rightarrow 0$ of Lie algebras making $C^\infty(M) \rightarrow \mathfrak{symp}(M)$ a central extension. Pull it back to get a central extension $\hat{\mathfrak{g}}$ of \mathfrak{g} , and the dual of this gives a G -equivariant map $M \rightarrow \hat{\mathfrak{g}}^*$.

The group version is based on $\text{Aut}(\mathcal{L}) \rightarrow \text{Symp}(M)$, where the automorphisms of \mathcal{L} may move the base but must preserve parallel transport. \square

- Let \mathbb{R}^{2n} act on itself by translation, preserving the standard symplectic form. Then the above construction discovers the Heisenberg group.
- Let $\text{Sp}(\mathbb{R}^{2n})$ act on $\mathbb{R}^{2n} \setminus \vec{0}$. This is a *double cover* of the minimal coadjoint orbit $(\mathbb{R}^{2n} \setminus \vec{0})/\pm$. If one tries to act on the Hermitian line bundle, one discovers the metaplectic group.
- Let LK act on ΩK . Then the above discovers that LK has a central extension \widehat{LK} , of which ΩK is a coadjoint orbit.

Central extensions of a group are related to elements of H^2 . If the group is compact connected, then that H^2 is also related to H^2 of its Lie algebra. So we get the weird situation that even though $H^2(\mathbb{R}^{2n}) = 0$, we have $H^2(\mathfrak{r}^{2n}) \cong H^2(\mathfrak{t}^{2n}) \cong H^2(\mathbb{T}^{2n}) \neq 0$, so \mathbb{R}^{2n} can have a central extension. The symplectic Lie algebra is semisimple, so has no central extensions, but the group is homotopic to $U(n)$ so has a double (or even \mathbb{Z} -) cover. Finally, $H^2(LK) \cong H^3(K) \cong \pi_3(K) \cong \mathbb{Z}$ for K simple and simply-connected, which gives a hint as to why LK should have a canonical central extension.

Morse-Bott theory on this manifold is a little weird, not so much because it's infinite-dimensional but because it's noncompact. Consider Morse theory on the punctured torus, using a function that goes to ∞ at the puncture. The Morse strata then form a figure 8, which is only a deformation retract of the punctured torus, rather than equal to it.

Theorem 10.4. ΩK deformation-retracts to the union $Gr = \coprod_{\lambda \in t_+} Gr_\circ^\lambda$ of the finite-dimensional Morse-Bott strata. Each Gr_\circ^λ is isomorphic to a complex vector bundle over a complex flag manifold K/K_λ , each closure is a projective variety, and the union is an "ind-scheme".

We heard already that dominant weights control representation theory. So how can we pull representations out of these Gr^λ ?

Baby case: $K = U(n)$, ΩK has \mathbb{Z} -many components. The minimum A -stratum on the k th component is isomorphic to $Gr_{k \bmod n}(\mathbb{C}^n)$. The homology of that manifold is $\binom{n}{k}$ -dimensional, which by amazing coincidence is also the dimension of the k th fundamental representation of $U(n)$!

Of course, that's also the cohomology of that manifold. But Gr^λ is singular in general, so these will differ, and which should we use? In general we won't want to use either, but the intersection homology, which we give a brief picture of.

The homology of a singular (or any) space is easy to think about geometrically, using cycles. The cohomology is just as easy if the space is smooth. The best smoothness we have available here is that each $Gr^\lambda = \coprod_{\mu \leq \lambda} Gr_\circ^\mu$ is stratified by smooth manifolds Gr_\circ^μ , so instead of thinking about arbitrary cycles, we think about cycles that "behave well" with respect to the stratification.

The cohomology of a (compact, oriented, and nicely) stratified space X is easy to describe using cycles C that are dimensionally transverse to the strata Y :

$$\dim C - (C \cap Y) \geq \dim X - \dim(Y = X \cap Y)$$

whereas to compute homology we don't need any condition:

$$\dim C - (C \cap Y) \geq 0.$$

For "intersection homology in middle perversity", we split the difference:

$$\dim C - (C \cap Y) \geq \frac{1}{2}(\dim X - \dim Y).$$

Call such C intersection homology chains, and make a complex with them, with the usual boundary as differential, to define $IH(X)$. It is naturally a module over $H^*(X)$ by a sort of cap product, but this isn't so useful since $H^*(X)$ can be so nasty.

Theorem 10.5 (Geometric Satake correspondence). *There is a natural action of ${}^L G$ on $IH(Gr^\lambda)$, making the latter into the irrep of ${}^L G$ with highest weight λ . Here ${}^L G$ is the Langlands dual group of G , whose weight lattice is the \mathbb{Z} -dual of G 's coweight lattice, and vice versa.*

The proof of this (first approximated by Ginzburg, now spread out over papers of Ginzburg, Lusztig, and Mirkovič-Vilonen) is rather indirect; they make a category that looks like the representations of some group, in that it has tensor products (the hard part) and a forgetful functor to \mathbf{Vec} , whose simple objects are the $\mathrm{IH}(\mathrm{Gr}^\lambda)$. Then the Tannaka reconstruction theorem says that this category *is* the representations of some group. Which one? With not much work, they relate its weight lattice with G 's coweight lattice, finishing the identification.

- Theorem 10.6.** (1) *(Mirkovič-Vilonen '99) If one does Morse theory on the singular space Gr^λ , using a component X_\cdot of the \mathbb{T} moment map, the Morse strata are reducible varieties, and their components give a basis of $\mathrm{IH}(\mathrm{Gr}^\lambda)$.*
- (2) *(Jared Anderson '03) One can use these cycles to compute weight multiplicities and tensor products of representations. It is even enough to know just their moment polytopes, "M-V polytopes".*
- (3) *(Kamnitzer '05) The polytopes are all distinct, and there is a simple characterization of them, bypassing all the infinite-dimensional geometry.*

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