# $K$-ORBITS ON $G/B$
# NOTES FOR A TOPICS COURSE, FALL 2012

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## 1. B-ORBITS ON G/P

For most of this course, we'll be considering groups as algebraic varieties over $\mathbb{C}$, or some other algebraically closed field of large enough characteristic (including 0). There will be a brief interlude later where it will be convenient to work over $\mathbb{Z}$ so as to be able to reduce mod $p$, but it will pass. Our references for algebraic groups are [FH, Hu].

As such, an **affine algebraic group** is a subgroup $G \leq GL_n(\mathbb{C})$ defined by polynomial equations, such as $SL_n, O_n, SO_n, Sp_n$, in particular *not* involving complex conjugation as would be necessary to define $GL_n(\mathbb{R}), U(n)$ or even worse, things like $GL_n(\mathbb{R})_+, GL_n(\mathbb{Z})$. We will always assume that $G$ is connected unless explicitly stated otherwise.

A **Borel subgroup** $B \leq G$ is a maximal connected solvable subgroup. They are all $G$-conjugate, in an essentially unique way (meaning, $N(B) = B$), and inside $GL_n$ the standard one is the upper triangular matrices. One of the principal reasons for their importance is this rather easy theorem:

**Theorem 1.1** (Borel)**.** *Let a solvable group $B$ act on a nonempty complete variety $X$. Then $X^B$ is also nonempty.*

*If moreover $G$ acts, then $X$ must contain a $G$-orbit isomorphic to $G/P$, where $P \geq B$.*

*In particular, a compact homogeneous space for $G$ must be $G$-isomorphic to some $G/P$.*

A subgroup $P$ containing $B$ is called a **parabolic subgroup**, about which more anon.

*Proof.* Let $B = B_0 > B_1 > B_2 > \ldots > B_n = 1$ be subgroups, each normal in the previous. Then each $B_i$ acts on $X^{B_{i+1}}$, and the action descends to one of $B_i/B_{i+1}$ on $X^{B_{i+1}}$.

Since $B$ is solvable, we can get each $B_i/B_{i+1}$ abelian. Extend the chain further so the quotients are 1-dimensional. If Borel's theorem holds for 1-dimensional groups, then by induction backwards from $B_{n-1}/1$ acting on $X = X^{B_n}$, each of these spaces and finally $X^B$ itself are nonempty.

It remains to consider the case that $B$ is 1-dimensional, and connected hence abelian. Assume there are no fixed points. Then any orbit $B \cdot x \subseteq X$ is irreducible and 1-dimensional, so $\overline{B \cdot x} \setminus (B \cdot x)$ is 0-dimensional, which would make its elements be $B$-fixed points, contradiction. So $B \cdot x \cong B/Stab(x)$ is closed hence projective. But $B/Stab(x)$ is again an affine algebraic group, so if also projective it's a point, contradiction.

Now imagine $G$ acts on $X$, and let $x \in X^B$ be a $B$-fixed point. Then $G \cdot x \cong G/\mathrm{Stab}(x)$ where $\mathrm{Stab}(x) \geq B$. □

If $G = GL_n$ and $B$ is standard Borel, then each $P$ is the block upper triangular matrices for a given decomposition $n = n_1 + \ldots + n_k$ into blocks, and $G/P$ is the **partial flag manifold** of chains $(0 < V^{n_1} < V^{n_1+n_2} < \ldots < \mathbb{C}^n)$ of subspaces with those subquotient dimensions.

A **maximal torus** or **Cartan subgroup** $T$ is a maximal subgroup isomorphic[1] to $(\mathbb{C}^\times)^k$.

**Theorem 1.2.** *Let $T$ act on a scheme $X$ of finite type (e.g. a subvariety of projective space). Then there are only finitely many subgroups $S \leq T$ occurring as stabilizers.*

Every Borel $B$ contains a maximal torus $T$ of $G$, unique up to $B$-conjugacy, whereas every $T$ is contained in $|W|$ many Borel subgroups. (Note that if we were working with real Lie groups, e.g. compact ones, rather than algebraic groups, then $T = B$ and this isn't true.)

More specifically, let $\Delta$ denote the set of **roots** of $G$, meaning the nonzero weights of $T$'s action on $\mathfrak{g} \otimes \overline{\mathbb{k}}$. Then each $\mathfrak{b}$ is determined by the half of $\Delta$ that it uses, called a **positive system**. The **unipotent subgroup** $N = [B, B]$ uses only those roots, without $T$, and $B = T \ltimes N$.

Given a pair $(B \geq T)$, let $\Delta_+ \subseteq \Delta$ denote the system of positive roots in $\mathfrak{b}$, and let $\Delta_1 \subseteq \Delta_+$ denote the **simple roots**, that aren't sums of other positive roots. (They turn out to be a basis for the **root lattice** that $\Delta$ spans, so one could let $\Delta_k \subseteq \Delta$ denote the elements that have total coefficient $k$ in this basis.)

Given a parabolic $P \geq B$, we can ask for which $\alpha \in \Delta_1$ does $-\alpha$ appear as weights in $\mathfrak{p}$, giving a correspondence between {parabolics containing $B$} and the power set of $\Delta_1$.

**Lemma 1.3.** *The exponential map $\mathfrak{n} \to N$ is a $T$-equivariant algebraic isomorphism, once one inverts small primes.*

*Proof.* It's enough to check this for $G = GL_n$, because we can extend $G$'s Borel subgroup to one of $GL_n$, prove the theorem there, and then restrict. Then $\mathfrak{n}$ is the strictly upper triangular matrices, and the exponential map is given by the power series, which has only finitely many terms, and only finitely many denominators. □

One less common Lie-theoretic fact we will need is this:

**Lemma 1.4.** *Let $B$ be a connected solvable Lie group, and $\mathfrak{t}, \mathfrak{t}'$ two Cartan subalgebras of $\mathfrak{b}$. Then they are conjugate by a unique element $\exp(\xi)$, $\xi \in \mathfrak{b}'$.*

*Proof.* *** **Humphreys?** *** □

[1]Over non-algebraically closed fields, one only asks for this isomorphism after passing to the algebraic closure. For example, inside $SL_2$ one can take either the diagonal matrices or $\{M : MM^T = 1\}$; over the reals these are isomorphic to $\mathbb{R}^\times$ and $S^1$ respectively, while over the complex numbers they are both isomorphic to $\mathbb{C}^\times$.

1.1. **The Bruhat decomposition and Schubert varieties.** Let $\mathcal{B}$ denote the space of Borel subgroups of G. If one picks a particular one, B, one can identify $\mathcal{B}$ with G/B. Pick also a $T \leq B$, giving a root system $\Delta$, split into B's roots $\Delta_+$ and the remainder $\Delta_- = -\Delta_+$.

Consider the T-fixed points on $\mathcal{B}$, i.e. the Borel subgroups C that T normalizes. But $N(C) = C$ so $C \geq T$, hence this set is the same as the $|W|$ many Borel subgroups $w \cdot B$ that contain T. The T-weights on the tangent space $T_{w \cdot B} G/B$ are $w \cdot \Delta_-$.

(Technically speaking one shouldn't write $w \cdot B$ for $w \in N(T)/T$, but $\widetilde{w} \cdot B$ where $\widetilde{w} \in N(T)$ is a lift. But the choice of lift doesn't matter because $T \leq B$, so we won't bother with the tildes.)

**Lemma 1.5.** *The multiplication map* $N_w \times N_{w_0 w} \to N$ *is a* T*-equivariant isomorphism of varieties.*

gotta define these

*Proof.* □

**Lemma 1.6.** *[The Bruhat decomposition: uniqueness] The* B*-orbit* $BwB/B$ *is* T*-invariant, isomorphic to an affine space of dimension* $\ell(w) := |\Delta_+ \cap w \cdot \Delta_-|$*, and has no other* T*-fixed points* $vB/B$ *in it,* $v \neq w$*.*

*In particular,* $G/B \supseteq \coprod_{v \in W} X_\circ^v$*, where* $X_\circ^v := BvB/B$ *is isomorphic to* $\mathbb{C}^{\ell(v)}$*.*

*Proof.* Since $wB/B$ is T-invariant, its B-orbit matches its N-orbit, which is isomorphic to $N/\mathrm{Stab}_N(wB/B)$. The Lie algebra of that is $\mathfrak{n} \cap$

$BwB/B \cong B/\mathrm{Stab}(wB/B)$, and $\mathrm{Stab}(wB/B) = B \cap w \cdot B \geq T$. Hence the B-orbit matches the N-orbit □

1.1.1. *Białynicki-Birula decompositions.* Let $\mathbb{G}_m$ act on a projective variety (or more generally, proper reduced scheme) X, and F be one of the finitely many components of $X^{\mathbb{G}_m}$. Define

$$X_\circ^F = \{x \in X : \lim_{t \to \infty} t \cdot x \in F\}$$

(where such "limits" exist by completeness and are unique by separatedness). Since any $\lim_{t \to 0} t \cdot x$ is $\mathbb{G}_m$-invariant,

$$X = \coprod_F X_\circ^F.$$

This is the algebro-geometric version of Morse-Bott theory:

**Theorem 1.7.** [BB73] *Let* X *be smooth and projective, with a* $\mathbb{G}_m$*-action. Then each* $X_\circ^F$ *is a vector bundle over* F*. The weights of* $\mathbb{G}_m$ *acting on* $T_f X$ *are independent of* $f \in F$*, and the relative dimension of* $X_\circ^F/F$ *is the number of positive weights.*

Pick now a circle $S : \mathbb{G}_m \to T$ such that each of the S-weights on $\mathfrak{n}$ is positive.

B-orbits, Białynicki-Birula definition, Morse theory definition, transversality of Bruhat and opposite Bruhat, **Kazhdan-Lusztig variety** $X_{w\circ}^v := X_w \cap X_\circ^v$, so named because of lemma 2.2 to come

1.1.2. *Stratifications generated by a divisor.* A **stratification of** $M$ **by closed subvarieties** is a collection $\mathcal{Y} \ni M$ of closed irreducible subvarieties $Y$, such that the intersection $Y_1 \cap Y_2$ of any two of them is (as a set) a union $\bigcup Y_i$ of finitely many others. For example, if $G$ is a connected group acting on $M$, then the $G$-orbit closures form a stratification. For sub-example, the Schubert varieties $\{X_w\}$ in $G/B$ form a stratification, where the group is $B_-$. If $\mathcal{Y} \subseteq \mathcal{Y}'$, call the first **coarser** and the second **finer**.

Given a collection $C$ of subvarieties of $M$, there is a unique coarsest stratification $\mathcal{Y}$ **generated** by the collection, which contains $C$ but also all the components of intersections of elements of $C$, and so on. More generally, one could start with a collection of reducible subschemes, let $C$ be the set of their irreducible components, and generate $\mathcal{Y}$ from there.

**Theorem 1.8.** *The Bruhat decomposition of* $X^v_\circ$ *is generated by the divisor* $X^v_\circ \setminus X^\circ_1$*, i.e. by the irreducible divisors* $\{X^v_{r_\alpha \circ}\}_{r_\alpha \leq v}$.

*Proof.* Since the $\{X_w\}$ already form a stratification, and their intersections with $X^v_\circ$ are irreducible, the $\{X_w \cap X^v_\circ\}$ also form a stratification, which therefore contains the stratification generated by $\{X^v_{r_\alpha \circ}\}_{r_\alpha \leq v}$. It remains to show that each $X_w \cap X^v_\circ$, $w \leq v$ actually occurs in this coarsest stratification $\mathcal{Y}$.

Assume that each Kazhdan-Lusztig subvariety $X_w \cap X^v_\circ$ occurs in $\mathcal{Y}$ when $\ell(w) \leq k$. Plainly this is true for $k = 0$, but we have also been given it for $k = 1$. Then we need a combinatorial lemma from [BGG73], that each $w \in W$ with $\ell(w) > 1$ is a Bruhat cover of at least two elements $w_1, w_2$. (Note that this is not true in the Bruhat order on $W/W_P$.)

By induction, $X_{w_1} \cap X^v_\circ$ and $X_{w_2} \cap X^v_\circ$ are in $\mathcal{Y}$. Their intersection is of lower dimension than either, and includes $X_w \cap X^v_\circ$ of codimension 1, hence includes it as a component. Therefore $X_w \cap X^v_\circ \in \mathcal{Y}$. ☐

1.2. **The (right) Bott-Samelson crank.** Given a simple root $\alpha$, we have a projection

$$\pi_\alpha : G/B \twoheadrightarrow G/P_\alpha$$

where $P_\alpha$ is the **minimal parabolic** such that $\mathfrak{p}_\alpha/\mathfrak{b}$ is 1-dimensional with weight $-\alpha$. This is a bundle with fiber $P_\alpha/B \cong \mathbb{P}^1$.

Given a space $Z$ with a map to $G/B$, define the **Bott-Samelson crank** $Z^\alpha\mathbf{BS}$ by the pullback diagram

$$\begin{array}{ccccc} Z^\alpha\mathbf{BS} & & \longrightarrow & & G/B \\ \downarrow & & & & \downarrow \pi_\alpha. \\ Z & \longrightarrow & G/B & \xrightarrow{\pi_\alpha} & G/P_\alpha \end{array}$$

This space is a $\mathbb{P}^1$-bundle over $Z$, and again has a map to $G/B$. Moreover, if the map $Z \to G/B$ is $H$-equivariant with respect to some subgroup $H \leq G$, so is $Z^\alpha\mathbf{BS} \to G/B$.

If $Q$ is a word $\alpha_1 \ldots \alpha_{|Q|}$ in the simple roots, we can turn the crank many times, defining $Z^Q\mathbf{BS}$ as $(\cdots(Z^{\alpha_1}\mathbf{BS})^{\alpha_2}\mathbf{BS})\cdots)^{\alpha_{|Q|}}\mathbf{BS}$. The most familiar case is that $Z$ is the basepoint $B/B \in G/B$, and $Z^Q\mathbf{BS}$ is a **Bott-Samelson manifold**, whose image is therefore a closed, irreducible, $B$-invariant subvariety of $G/B$, hence of the form $X^v$.

Define the **Demazure product** of $Q$ to be this $v \in W$, a variant on the product of the simple reflections $r_{\alpha_1} \cdots r_{\alpha_{|Q|}}$. Unlike the ordinary product, this definition is plainly monotonic (in Bruhat order) under increasing the word. From this geometric definition, one can see many equivalent combinatorial definitions, e.g. the unique Bruhat-maximum of the ordinary product of all subwords.

One inductive way to study Bott-Samelson cranks is to note that if $Z \to G/B$ has image $Y \hookrightarrow G/B$, then map $Z^\alpha G/B \to G/B$ factors through $Y^\alpha G/B$.

**Theorem 1.9.** *The following are equivalent:*

- $Q$ *is a reduced word for* $v$,
- *the map* $(B/B)^Q\mathbf{BS} \twoheadrightarrow X^v$ *is generically finite,*
- *the map* $(B/B)^Q\mathbf{BS} \twoheadrightarrow X^v$ *is an isomorphism over* $X^v_\circ$.

*Proof.* Let $Q'$ be $Q$ minus its last letter, hence reduced if $Q$ is. *** **and then...** *** □

1.2.1. *The left Bott-Samelson crank.* One can do these computations up inside $G$, instead of on $G/B$, where the map $X \to G/B$ is replaced by its pullback $\widetilde{X} \to G$, necessarily right-$B$-equivariant. Then the right Bott-Samelson crank takes

$$\widetilde{X} \xrightarrow{f} G \qquad \mapsto \qquad \widetilde{X} \times^B P_\alpha \to G$$

where the latter map is given by $[x, p] \mapsto f(x)p$.

In the case that $f : \widetilde{X} \to G$ is also *left*-$B$-equivariant, discussed above, then one has a completely symmetric definition of a left Bott-Samelson crank $P_\alpha \times^B \widetilde{X} \to G$, which works perfectly well down on $G/B$ as well, $P_\alpha \times^B X \to G/B$. We won't find this that useful as our eventual interest is in $K$-orbit closures where $K \not\geq B$, so we won't have this left-$B$-equivariance.

## 2. Local structure of Schubert varieties

Since $X_w$ is $B_-$-invariant, its singularity type is constant along each $B_-$-orbit $X^\circ_v$, $v \geq w$. So it is enough to study it in the neighborhood of each fixed point $vB/B$.

**Exercise 2.1.** *There is a unique* $T$*-invariant open affine neighborhood in* $G/B$ *containing* $vB/B$*, namely* $v \cdot X^\circ_1$.

**Lemma 2.2.** [KL] *One can factor* $v \cdot X^\circ_1$ *so as to make the following pullback diagrams:*

$$\begin{array}{ccc} v \cdot X^\circ_1 & \cong & X^\circ_v \times X^v_\circ \\ \uparrow & & \uparrow \\ X_w \cap (v \cdot X^\circ_1) & \cong & X^\circ_v \times (X_w \cap X^v_\circ) \end{array}$$

This says that all the action is contained in the Kazhdan-Lusztig variety $X^v_{w\circ} := X_w \cap X^v_\circ$. Luckily we have a good handle on these:

**Theorem 2.3.** *Let* $Q$ *be a reduced word for* $v$*. Fix isomorphisms of each simple root subgroup with* $SL_2$*, compatible with the choices of* $T, B$.

*Then we have* **Bott-Samelson coordinates**

$$\beta_Q : \mathbb{A}^{|Q|} \cong X^v_\circ$$
$$(c_1, \ldots, c_{|Q|}) \mapsto \left( \prod \widetilde{r}_{\alpha_i} e_i(c_i) \right) B/B.$$

*Proof.* Recall that if $Q$ is gives us a parametrization , then theorem 1.9 □

Let $I_w^Q$ denote the polynomial ideal defining the subvariety $\beta_Q^{-1}(X_{w_\circ}^v) \subseteq \mathbb{A}^{|Q|}$. Given any ideal $J \in \mathbb{k}[c_1, \ldots, c_{|Q|}]$, we can define $\mathtt{lex\text{-}init}(J)$ as the linear span of the lexicographically first terms of each $p \in J$. This is again an ideal, an example of a "Gröbner degeneration" of J.

**Theorem 2.4.** [K, §7.3] *Let* $Q$ *be a reduced word for* $v \geq w$. *Then* $\mathtt{lex\text{-}init}(I_w^Q)$ *is generated by* squarefree *monomials, so vanishes on a reduced union of coordinate subspaces.*

Ideals I generated by squarefree monomials are called **Stanley-Reisner** ideals, and correspond naturally to simplicial complexes $\Delta$, where $F \subseteq \{1, \ldots, n\}$ is a face of $\Delta$ if I vanishes on $\mathbb{A}^F$.

Maybe there'll be time to discuss the Frobenius splitting argument that gives the quickest proof of this statement.

It remains to describe the simplicial complexes.

### 2.1. **Subword complexes.**

Given any word Q (not necessarily reduced), let $\Delta^{|Q|-1}$ denote the simplex with vertices Q, and define a discontinuous function $\delta$ on it by

$$\begin{array}{rcl} \{\text{open faces of } \Delta^{|Q|-1}\} & \to & W \\ F \subseteq Q & \mapsto & \text{the Demazure product of } Q \setminus F. \end{array}$$

Define $\Delta(Q, w)^\circ := \delta^{-1}(w)$, considered as a union of open faces in the simplex, and $\Delta(Q, w) := \delta^{-1}(\{w' \geq w\})$. So

$$\Delta^{|Q|-1} = \Delta(Q, 1) = \coprod_w \Delta(Q, w)^\circ$$

and as we shall see, we can think of this as a Bruhat decomposition of the simplex.

**Theorem 2.5.** [KM04] $\Delta(Q, w)$ *is a simplicial complex, called a* **subword complex**. *If nonempty, it is homeomorphic to a ball or sphere of dimension* $|Q| - \ell(w)$, *with interior* $\Delta(Q, w)^\circ$. *(So it is a sphere exactly if* $Q$ *is nonreduced and* $w$ *is its Demazure product.)*

This is actually very easy to prove by induction on $|Q|$, using the concept of a "vertex decomposition"; breaking $\Delta(Q, w)$ into the subcomplexes that use the last letter or don't gives a picture of $\Delta(Q, w)$ as union of two balls glued together along a ball or sphere on the boundary of each. Ezra Miller and I invented these complexes (and studied them completely combinatorially in [KM04]) some years before I figured out that they are exactly what one needs to study general Kazhdan-Lusztig varieties.

We can now state the complete version of theorem 2.4; $\mathtt{lex\text{-}init}(I_w^Q) = SR(\Delta(Q, w))$, where $SR$ denotes the Stanley-Reisner ideal.

### 2.2. **Geometric consequences.**

Let $p \in M$ be a point (not necessarily closed) of codimension $d$, and $U \ni p$ an affine neighborhood. If we chop $U$ with a function that isn't a zero divisor (i.e. the hypersurface $f = 0$ cuts down *every* component of U), then we can consider $p$ in a new space, now of codimension $d - 1$. If we can uncover $p$ or some schemy thickening of it, using a succession of non-zero-divisors, we say X is **Cohen-Macaulay at** p. One reference is [Ei, §??].

Non-example: Let X be the union of two planes in 4-space, meeting at the origin p. If we chop this using a generic linear subspace, it gives two lines, neither vanishing at

$p$. Considering the limit where the lines do both go through $p$, we get a union of two lines with an embedded point at $p$. Then the next linear subspace cuts down the two lines, but not the enbedded point, i.e. no non-zero-divisor is available. So this $X$ is not Cohen-Macaulay at this $p$.

If $X$ is Cohen-Macaulay at every point, it is just called Cohen-Macaulay. This property is open on the base, in flat proper families, and such properties are called **semicontinuous**. In particular, given a flat family $F \to \mathbb{A}^1$, equivariant with respect to dilating the $\mathbb{A}^1$, if $F_0$ is Cohen-Macaulay then each $F_t$ is too.

Part of the utility of Cohen-Macaulayness is its multiple interpretations. Another one is in terms of the vanishing of "local cohomology" groups.

**Theorem 2.6.** [Ho] *If $\Delta$ is a shellable simplicial complex, then $SR(\Delta)$ is Cohen-Macaulay.*

*Proof sketch.* Local cohomology satisfies a Mayer-Vietoris principle implying that if $A, B$ are C-M of dimension $n$, and $A \cap B$ is C-M of dimension $n-1$, then $A \cup B$ is C-M (of dimension $n$). Then one does induction with the shelling. □

**Corollary 2.7.** [R] *Schubert varieties are Cohen-Macaulay.*

*Proof.* Crossing with a vector space doesn't change Cohen-Macaulayness, so by the Kazhdan-Lusztig lemma it's enough to show that Kazhdan-Lusztig varieties are C-M. Since they Gröbner-degenerate to subword complexes, which are shellable, Hochster's theorem and semicontinuity do the rest. □

A variety $X$ is **normal** if any **finite map** $X' \twoheadrightarrow X$ (proper with finite fibers) that is generically an isomorphism, is an isomorphism everywhere. Non-example: if $X$ is a nondisjoint union of components, let $X'$ be the disjoint union. So the union of two planes discussed above is neither Cohen-Macaulay nor normal. However, the union $\{xy = 0\}$ of two lines in the plane *is* Cohen-Macaulay, while not normal. (It is also possible to be normal but not Cohen-Macaulay.)

**Theorem 2.8.** [Ei, R1+S2 and Sn]

1. *If $X$ is normal, then its singular locus has codimension $> 1$; one says that $X$ is R1.*
2. *If $X$ is Cohen-Macaulay and R1, then it is normal.*

(C-M is equivalent to the strong "Serre condition" $S_n$; under a weaker condition $S_2$ the above becomes if-and-only-if.)

**Theorem 2.9.** [R] *Schubert varieties are normal.*

*Proof.* If $X_w$ isn't normal, it's because it's singular in codimension 1, along some $X_{w'}, w' \gtrdot w$. By the Kazhdan-Lusztig lemma, the curve $X^w_{w'\circ}$ is singular at the point $w'B/B$. But we know $X^w_{w'\circ}$ "degenerates" to the Stanley-Reisner scheme of a point, namely $\mathbb{A}^1$ (scare quotes because the degeneration is actually a constant family). □

## 3. SYMMETRIC SUBGROUPS

### 3.1. Borel-de Siebenthal theory.

## 3.2. **Satake diagrams.**

**Lemma 3.1.** $\theta$ *acts on the Dynkin diagram.*

*Proof.* Let $B \in \mathcal{B}$. We can identify the set of simple roots with the B-irreps in $\mathfrak{b}'/\mathfrak{b}''$, giving a necessarily trivial covering space of the simply-connected space $\mathcal{B}$. There is a connected set of $g$ such that $g \cdot \theta \cdot B = B$, each of which gives an involution on this set of simple roots, continuously (hence constantly) in $g$. Since $g\theta$ is an automorphism of B, and we can determine the angles between roots from B's group structure, the involution preserves the Dynkin diagram. $\square$

This involution is almost, but not quite enough, information to determine $\theta$ up to conjugacy. It turns out that the fixed points still come in two types: "imaginary" and not.

We assume a result from later (corollary 4.11): there exist tori T such that $\theta \cdot T = T$. Then $\theta$ acts also on T's associated root system, and we call a root $\alpha$ **imaginary** if $\theta(\alpha) = \alpha$, and let $\Delta_{im}, \Delta_{\not{im}}$ denote the simple roots that are or aren't imaginary.

There is a subtle but important point here. If we fix a $B \geq T$, it is usually *not* $\theta$-stable, so the action of $\theta$ on T's $\Delta$ does *not* contain the action of $\theta$ defined above on $\Delta_1$, unless B happens to be $\theta$-stable. Beware!

Since $\theta$ acts on $\mathfrak{t}$ with eigenvalues $\pm 1$, and we define the **split rank** of a $\theta$-stable torus as the number of $-1$ eigenvalues.

**Example 3.2.** $SL_2(\mathbb{R})$

**Lemma 3.3.** *Let* $A, B$ *be matrices in* $M_n(\mathbb{N})$ *with* $AB = I_n$*. Then* $A, B$ *are inverse permutation matrices.*

*Proof.* Since $A \in M_n(\mathbb{R}_{\geq 0})$ and has no zero rows, for any $\mathbb{R}_{\geq 0}$-valued vector $\vec{v}$, $|A\vec{v}|_1 \geq |\vec{v}|_1$ (the $L^1$ norm, namely, the sum of the entries).

Hence $|\vec{v}|_1 = |AB\vec{v}_1| \geq |B\vec{v}_1| \geq |\vec{v}|_1$. Thus $|B\vec{v}|_1 = |\vec{v}|_1$. Applied to the basis vectors, we learn that B's columns all have $L^1$-norm 1. Since $B \in M_n(\mathbb{N})$ we learn that each column has one 1 and the rest 0s. Being invertible, it's a permutation matrix. $\square$

**Theorem 3.4** (Satake)**.** *Let* T *be a* $\theta$*-stable torus. Call a positive system* $\Delta_+ \subseteq \Delta$ **Satake** *if for each simple root* $\beta \in \Delta_{\not{im}}$*, we have* $\theta(\beta) < 0$*. In (2)-(5) we assume a Satake system is chosen.*

(1) *There exist Satake systems for* $(G, \theta, T)$*. More specifically, if* $\Delta_{im}$ *denotes the imaginary simple roots for some positive system* $\Delta_+$*, then there exists a Satake system* $\Delta'_+$ *such that* $\Delta'_{im} \supseteq \Delta_{im}$*.*
(2) *If* $\alpha \in \Delta_1$ *is not imaginary, then* $\theta(\alpha) = -\alpha' +$ *a combination of imaginary simple roots, and the map* $\alpha \mapsto \alpha'$ *defines an involution on the non-imaginary roots (*not *usually the one from lemma 3.1, and in particular,* not *a Dynkin diagram automorphism!).*
(3) *The union of* $\Delta_{im}$ *and the set* $\{\alpha - \alpha'\}$*, where* $(\alpha, \alpha')$ *varies over the 2-cycles just induced on the non-imaginary simple roots, is a* $\mathbb{Z}$*-basis for the* 1*-eigenspace of* $\theta$ *on the root lattice.*
*If* $\theta(\alpha) = -\alpha' + \gamma$ *and* $\theta(\alpha') = -\alpha + \gamma'$*, then* $\alpha + \alpha' + \frac{1}{2}(\gamma + \gamma')$ *is negated by* $\theta$*, and these vectors are a* $\mathbb{Q}$*-basis for the* $(-1)$*-eigenspace of* $\theta$ *on the root lattice.*
(4) *If* $\beta \in \Delta_+$ *is not in the span of* $\Delta_{im}$*, then* $\theta(\beta) < 0$*, and in particular* $\theta(\beta) \neq \beta$*.*
(5) *If* $\theta(\beta) = -\beta$*, call* $\beta$ *a* **real** *root. Any real root is a combination of real simple roots.*

(6) *If we color the imaginary roots black, the remainder white, and indicate the involution on the white vertices, the resulting* **Satake diagram of** $(G, \theta, T)$ *does not depend on the choice of Satake system. The split rank is the number of white orbits.*
(7) *Let* $T$ *be a* **maximally split** $\theta$*-stable torus (i.e. having maximum split rank). Then the choice of* **Satake diagram of** $(G, \theta)$ *does not depend on* $T$*, and determines* $\theta$ *up to conjugacy. The induced involution on the simple roots is the one given by lemma 3.1.*

All cases are drawn on the next couple pages.

*Proof.* (1) Start with a positive system $\Delta_+$, so $\theta \cdot \Delta_+ = w \cdot \Delta_+$ for a unique $w \in W$. Let $m$ be the number of simple roots $\alpha$ such that $\theta(\alpha) \in \Delta_+ \setminus \alpha$. If $m = 0$, we're done; otherwise pick one, $\alpha$. Let $\Delta'_+ = r_\alpha \Delta_+ = \Delta_+ \setminus \alpha \cup \{-\alpha\}$.

If $\theta(\alpha) \in \Delta_1$, then $-\alpha, \theta(\alpha)$ are simple roots in $\Delta'_+$ taken negative by $\theta$, so $m$ has decreased by 2.

Otherwise applying $\theta$ to the $\Delta'_+$-simple root $-\alpha$, we get $\theta \cdot (-\alpha) = -\theta \cdot \alpha \in \Delta'_-$. So $m$ has decreased by 1. Either way we can continue until $m = 0$.

(2) If we order the basis $\Delta_1$ with the non-imaginary roots first and the imaginary roots second, the matrix of $\theta$ looks like $\begin{pmatrix} A & 0 \\ B & \mathrm{Id} \end{pmatrix}$, where $A, B$ have entries in $-\mathbb{N}$. Applying lemma 3.3 to $-A$, we learn that $-A$ is a permutation matrix, so the the claimed involution on the non-imaginary roots.

(3) First we claim that $\alpha - \alpha'$ is $\theta$-invariant. By the formula just computed, $\theta$ of it is $-\alpha' + \alpha + \gamma$, where $\gamma$ is in the span of the simple imaginary roots. Applying $\theta$ again, we learn $\alpha - \alpha' = \alpha - \alpha' + 2\gamma$, so $\gamma = 0$. Since the simple imaginary roots are part of a $\mathbb{Z}$-basis for the root lattice, we can mod them out. Now the action is a direct sum of actions on $\mathbb{Z}^2$, each spanned by a pair $\alpha, \alpha'$, and within there $\alpha - \alpha'$ $\mathbb{Z}$-generates the 1-eigenspace.

It is easy to see that $\alpha + \alpha' + \frac{1}{2}(\gamma + \gamma')$ is negated by $\theta$, and that these are linearly independent (even modulo $\Delta_{im}$). Together, these are the right number of elements for a $\mathbb{Q}$-basis of $T^*$.

(4) Write $\beta = \sum_{\alpha \in \Delta_{\not{im}}} n_\alpha \alpha + \gamma$, where $\gamma$ is in the $\mathbb{N}$-span of $\Delta_{im}$. Then all $n_\alpha \geq 0$, and some $n_\alpha > 0$. Hence $\theta(\beta) = \sum_{\alpha \in \Delta_{\not{im}}} n_\alpha(-\alpha') + \gamma'$, where $\gamma'$ is in the $\mathbb{Z}$-span of $\Delta_{im}$, and $\alpha'$ is the simple root corresponded with $\alpha$ in (2). So some coefficient of $\theta(\beta)$ is strictly negative.

(5) The subsystem $\Lambda := \Delta \cap (T^*)^\theta$ of $\theta$-invariant roots doesn't depend on the choice of Satake system. By (4), if $\Delta_+$ is Satake then $\Lambda$ is generated by $\Lambda \cap \Delta_1$, hence determines a Dynkin subdiagram isomorphic to $\Lambda$'s. (Though we don't know yet that the complements are isomorphic!)

Note that $\Lambda$ doesn't usually generate $(T^*)^\theta$.

...why not?

(6) These should live in the Borels in the open $K$-orbit...

□

3.2.1. *An example.* Let $G = GL(n), \theta(M) = JM^{-T}J^{-1}$, where $J$ is the permutation matrix for $w_0$. Hence $K = O(n)$. If $n = 2$, the $\theta$-stable Cartan subalgebras are the scalars plus

$$\mathbb{C} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbb{C} \cdot \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}, b, c \neq 0$$

with split ranks $0, 1$ respectively, called the **compact** and **split** tori. The weight vectors of these tori are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ for the first, and $\begin{bmatrix} \sqrt{b} \\ \pm\sqrt{c} \end{bmatrix}$ for the second. The action of $\theta$ is trivial on the first torus' $\Delta$ (i.e. an imaginary root), and nontrivial on the second's (i.e. not an imaginary root).

To make $\theta$-stable tori for larger $n$, inside the $GL(2)$ corresponding to each orbit of $w_0$ (other than the fixed point, if $n$ is odd) we pick one of these two 2-tori. The split rank $r$ is then the number of times we choose the latter kind of torus.

We can correspond $\Delta$ with the set of ordered pairs of $T$-weight vectors, and a choice of Borel containing $T$ is an ordering on the weight vectors. If we think of $\theta$ as switching the two weight vectors in the split case, and leaving the two weight vectors alone in the compact case, then a Satake-adapted ordering of the weight vectors has the all the compact vectors in the middle, with the split vectors forming $r$ nested pairs around the outside.

If $n = 2r$, then all the vertices are white, and the involution on the diagram is reversal, leaving a fixed white vertex in the middle. Otherwise $n > 2r$, and there are $n - 1 - 2r$ black vertices in the middle, surrounded by $r$ white vertices on the left matched by the involution to $r$ white vertices on the right. If there are two or more black vertices, the involution is not a Dynkin diagram automorphism.

3.3. **Connection to real forms.** Let $G_{\mathbb{R}}$ be a connected real algebraic group, with maximal compact subgroup $K_{\mathbb{R}}$, the fixed points of a Cartan involution $\theta$. Extend this to an involution $\theta$ of $G = G_{\mathbb{C}}$, with fixed points $K = K_{\mathbb{C}}$. So $K_{\mathbb{R}}$ extends to a maximal compact $G_c$ of $G$.

If $G_{\mathbb{R}}$ is a complex group, regarded as a real one, then $G \cong G_{\mathbb{R}} \times G_{\mathbb{R}}$, with $\theta$ acting by switching the two factors. Let $g \mapsto \overline{g}$ denote the Cartan involution on $G_{\mathbb{R}}$, fixing the maximal compact $K_{\mathbb{R}}$. Its complexification $K$ sits inside $G$ as $\{(g, \overline{g})\}$.

3.4. **The Matsuki correspondence: statement.** Let $B \leq G$ be a Borel subgroup.

**Theorem 3.5.** (1) [Ma] *There is an anticorrespondence of the posets* $K\backslash G/B$ *and* $G_{\mathbb{R}}\backslash G/B$.

(2) *Let* $f : M \to \mathbb{R}$ *be a smooth function on a compact manifold, and* $C$ *the set of components of* $f$*'s critical set. Assume* $C$ *is finite (e.g. if* $f$ *is real algebraic), and that the Morse decompositions of* $f$ *and* $-f$ *are both stratifications. Then these Morse posets are antiisomorphic, and both correspond to* $C$.

(3) [Ma] *Let* $G_c \cdot \lambda \subseteq \mathfrak{g}_c^*$ *be a generic* $G_c$*-orbit, uniquely* $G_c$*-isomorphic to* $G/B$. *Let* $\Phi : G/B \to \mathfrak{k}_{\mathbb{R}}^*$ *be the* $K_{\mathbb{R}}$*-equivariant composite*

$$G/B \cong G_c \cdot \lambda \hookrightarrow \mathfrak{g}_c^* \twoheadrightarrow \mathfrak{k}_{\mathbb{R}}^*.$$

*Then* $|\Phi|^2$ *satisfies the conditions of part (2), and the Morse strata of* $|\Phi|^2, -|\Phi|^2$ *are the* $K$*-orbits and* $G_{\mathbb{R}}$*-orbits, respectively.*

3.5. $\mathsf{GL}(n)$ **examples.** Here the space $\mathcal{B}$ can be identified with the space of flags in $\mathbb{C}^n$, and an obvious invariant of the $K$-orbit is the relative Schubert position of the two flags $F$ and $\theta \cdot F$. The coadjoint orbit $G_c \cdot \lambda$ can be thought of as the space of Hermitian matrices of a fixed spectrum $\lambda$ without repeated eigenvalues, where its isomorphism to $\mathsf{Flags}(\mathbb{C}^n)$ defines $F_d$ as the sum of the first $d$ eigenlines.

Examples:

(1)
(2)
(3) $G_{\mathbb{R}} = U(p, q)$. Then $G = GL(p + q)$, $K = GL(p) \times GL(q)$, the centralizer of an involution $J$ on $V = \mathbb{C}^{p+q}$. For an invariant of the $G_{\mathbb{R}}$-orbit, look at the signature of the form on each subspace $F_d$ in the flag. At each step, the dimensions of the positive and negative parts can increase by dimension 0 or 1, ending at $(p, q)$. Call this sequence $(p_i, q_i)$. If we pick $F_d^{\pm}$, we can split $V$ canonically as the orthogonal direct sum $F_d^+ \oplus F_d^- \oplus V_d'$, and $F_d$ likewise as $F_d^+ \oplus F_d^- \oplus R_d$. Since $V_d'$ is nondegenerate with signature $(p - p_d, q - q_d)$, and $R_d$ is isotropic in $V_d'$, we have $\dim R_d \leq \min(p - p_d, q - q_d)$. So

$$d \geq p_d + q_d \geq d - (p - p_d), d - (q - q_d).$$

For the invariant of the $K$-orbit, look at $F_d \cap J \cdot F_d$, which is $J$-invariant and so $J$ acts on it with eigenvalues $\pm 1$, the total number being $\leq d$. In other words, $p_d = \dim(F_d \cap V_+)$, $q_d = \dim(F_d \cap V_-)$, where $V_\pm$ are the eigenspaces.

(Because we're in the special case that $K$ is a Levi subgroup, it's the intersection of two opposed parabolics, which turn out to be unique. Individually, their two orbit decompositions give the $(p_d)$, $(q_d)$ invariants separately.)

If we fix a $\pm$positive definite Hermitian form on $V_\pm$, we can define $G_{\mathbb{R}}$ as preserving the pseudoHermitian form on $V_+ \oplus V_-$.

Fix now an invariant $(p_i, q_i)_{i=0,\ldots,p+q}$. So this defines a $G_{\mathbb{R}}$-orbit and a $K$-orbit. In their intersection, we have flags $F$ such that
(a) on $F_d$, the pseudo-Hermitian form has signature $(p_d, q_d)$
(b) $\dim(F_d \cap V_+) = p_d$, $\dim(F_d \cap V_-) = q_d$.
Hence $F_d$ is the orthogonal direct sum of $F_d \cap V_+$, $F_d \cap V_-$, and its radical $R_d := F_d \cap F_d^\perp$. That is, $R_d$ is an isotropic space inside the nondegenerate space $\pi_+(R_d) \oplus \pi_-(R_d)$, where $\pi_\pm$ are the projections to $V_\pm$, and $\pi_\pm(R_d)$ is perpendicular to $F_d \cap V_\pm$.

To construct such an $F_d$, given $F_{d-1}$, there are several cases. If $p_{d-1} < p$, pick a line $L_+ \leq V_+ \cap \pi_+(R_d)^\perp$, and similarly an $L_-$ if $q_{d-1} < q$.
- – If $p_{d-1} = p$, then

we
- add on a new line from $V_+ \cap \pi_+(R_d)^\perp$ if $p_d > p_{d-1}$ but $q_d = q_{d-1}$,
- add on a new line from $V_- \cap \pi_-(R_d)^\perp$ if $p_d = p_{d-1}$ but $q_d > q_{d-1}$,
- a

The map $\Phi$ takes a Hermitian matrix $H$ to its $(p, q)$ diagonal blocks, and then to the sum of their norm-squares. The critical points of $\Phi$ itself are when $H$ is block diagonal.

Let's compute whether $H = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ is a critical point of $|\Phi|^2$. The tangent space $T_H G/B$ is the image of $\mathfrak{u}(p+q)$, and it's enough to work with $(\mathfrak{u}(p) \oplus \mathfrak{u}(q))^\perp =$

$\left\{\begin{pmatrix} 0 & D \\ -D^* & 0 \end{pmatrix}\right\}$, which give the tangent vectors $\begin{pmatrix} DB^* - BD^* & DC - AD \\ D^*A - CD^* & D^*B - B^*D \end{pmatrix}$. Then the directional derivative is

$$\mathrm{Tr}(ADB^* - ABD^* + D^*BC - B^*DC) = \mathrm{Im}\,\mathrm{Tr}(ADB^* + D^*BC).$$

Using the $U(p) \times U(q)$-invariance, we can assume $A, C$ are diagonal hence real.

## 4. K-ORBITS ON G/B

Let $\theta : G \to G$ be an involution, with $K := G^\theta$ the corresponding **symmetric subgroup**. We will always be concerned with the case that $G$ and $\theta$ are complex algebraic.

However, these originally arose in the study of "symmetric spaces", which are defined as Riemannian manifolds $M$ such that for each $p \in M$, there exists an isometry of $M$ fixing $p$ and acting as $-1$ on $T_pM$. The big theorem is that $M$ must be locally of the form $G/K$, where $G$ and $K$ are real Lie groups. The standard references are [He1, He2].

The automorphism group of $G$ is the semidirect product of the automorphism group of the Dynkin diagram with the inner automorphism group $G/Z(G)$. Hence if $G$'s diagram has no automorphisms, the only $\theta$s are inner, i.e. conjugation by some element $g$ of order 2 in $G/Z(G)$.

Examples:

- $G = K \times K$, and $\theta \cdot (a, b) = (b, a)$, so $G^\theta$ is the diagonal subgroup $K_\Delta$.
  On the Dynkin diagram, $\theta$ switches the two copies of $K$'s diagram inside $G$'s.
- $G = GL_n(\mathbb{C})$, $\theta \cdot M = (M^{-1})^T$, so $K = O_n(\mathbb{C})$. Here $\theta$ flips $G$'s diagram.
- Let $J$ be an invertible antisymmetric matrix, necessarily of even size $2n$.
  Let $G = GL_{2n}(\mathbb{C})$, and $\theta \cdot M = J(M^{-1})^T J^{-1}$. Then $K = Sp(\mathbb{C}^{2n}, J)$. Again, $\theta$ flips $G$'s diagram, showing that the action on the diagram doesn't tell one everything.
- Let $g \in G$ be of order 2 in $G/Z(G)$, and $\theta$ be $g$-conjugation. Then $K = C_G(g)$.
  In these inner cases the action on the Dynkin diagram is trivial.
- More specifically, let $G = GL(a + b)$ and $g = diag(1^a, (-1)^b)$.
  Then $K = GL(a) \times GL(b)$.

These include all the cases when $G = GL(n)$, *** **check** *** but there are many more.

### 4.1. Some examples.

The most classic examples of symmetric subgroups are the stabilizers inside $GL(n)$ of orthogonal or symplectic forms, and to study their orbits we have

**Lemma 4.1** (Witt). *Let $V$ be a finite-dimensional vector space with a symmetric or antisymmetric form $\langle,\rangle$,* possibly degenerate, *and $W, Y$ two subspaces, with an isomorphism $T : W \to Y$ that preserves the restriction of $\langle,\rangle$. Then there is an automorphism of $V$, preserving $\langle,\rangle$, extending $T$.*

**Corollary 4.2.** *Let $V$ be a finite-dimensional vector space with an orthogonal or symplectic form $\langle,\rangle$, and $(W_i), (Y_j)$ two flags such that $\dim(W_i \cap W_j^\perp) = \dim(Y_i \cap Y_j^\perp)$. Then there is an automorphism of $(V, \langle,\rangle)$ taking $W$ to $Y$.*

4.1.1. *Example:* $G = SL(3), K = SO(3)$. If we identify $SL(3)/B$ with flags $(0 < L < P < \mathbb{C}^3)$, then the orbit closures are the whole space, $D_1 := \{\mathrm{rank}\ L = 0\}$, $D_2 := \{\mathrm{rank}\ P = 1\}$, and the closed orbit $\{P = L^\perp\}$.

Since a rank 2 plane $P$ contains two isotropic lines, the map $BS^1D_1 \twoheadrightarrow m_1 \cdot D_1 = SL(3)/B$ is generically $2:1$. (It is $1:1$ over the subvariety $D_2$.) Whereas $m_2 \cdot D_1 = D_1$. The Dynkin diagram automorphism replaces $1 \leftrightarrow 2$ everywhere.

It is interesting to note that (in contrast to the Schubert variety situation) the intersection of the two K-divisors is not reduced. Lifting from $GL(3)/B$ to $GL(3) = \{[\vec{v}_1\vec{v}_2\vec{v}_3]\}$, the K-invariant equations defining $D_1, D_2$ are

$$|\vec{v}_1|^2 = 0, \qquad \det\begin{pmatrix} |\vec{v}_1|^2 & \vec{v}_1 \cdot \vec{v}_2 \\ \vec{v}_2 \cdot \vec{v}_1 & |\vec{v}_2|^2 \end{pmatrix} = 0.$$

Together, they imply $(\vec{v}_1 \cdot \vec{v}_2)^2 = 0$, which has $\vec{v}_1 \cdot \vec{v}_2 = 0$ properly in the radical.

4.1.2. *Example:* $G = GL(2n)$, $K = Sp(n)$. First, we need to describe $K\backslash G/B$. As usual $G/B$ can be identified with flags $(F_i)$ in $2n$-space, the Springer map with the matrix $\dim(F_i \cap F_j^\perp)$, and the twisted involutions with ordinary involutions.

More specifically, if $\tau \in S_{2n}$ is a permutation matrix, we can look at the set of $(F_i)$ such that $\dim(F_i \cap F_{2n-j}^\perp) = \#1\text{s in the bottom left } i \times j$ rectangle. Then

$$\begin{aligned}
\#1\text{s in bottom left } i \times j &= \dim(F_i \cap F_{2n-j}^\perp) \\
&= 2n - \dim(F_i \cap F_{2n-j}^\perp)^\perp \\
&= 2n - \dim(F_i^\perp + F_{2n-j}) \\
&= 2n - \left(\dim F_i^\perp + \dim F_{2n-j} - \dim(F_i^\perp \cap F_{2n-j})\right) \\
&= 2n - \left(2n - i + 2n - j - \dim(F_i^\perp \cap F_{2n-j})\right) \\
&= i + j - 2n + \dim(F_{2n-j} \cap F_i^\perp) \\
&= i + j - 2n + \#1\text{s in bottom left } (2n-j) \times (2n-i) \\
&= i - (2n - j - \#1\text{s in bottom left } (2n-j) \times (2n-i)) \\
&= i - \#1\text{s in bottom right } (2n-j) \times i \\
&= \#1\text{s in top right } j \times i
\end{aligned}$$

i.e. $\tau$ must be an involution.

Flags in the open orbit map to $w_0$, and in the closed orbit to $2i-1 \leftrightarrow 2i$. Call this permutation $\rho_{min}$.

If we apply this map to the T-invariant flag $\pi B/B$, we get the fixed-point-free involution $\rho := \pi^{-1} w_0 \pi$. Why only those?

$$\begin{aligned}
\#1\text{s in the top left } i \times i &= i - \#1\text{s in the bottom left } (2n-i) \times i \\
&= i - \dim(F_{2n-i} \cap F_{2n-i}^\perp) \\
&= 2n - i - \dim(F_{2n-i} \cap F_{2n-i}^\perp) + 2(i-n) \\
&= \text{rank } (F_{2n-i}) + 2(i-n) \\
&\equiv 0 \bmod 2
\end{aligned}$$

because the rank of a symplectic form on a vector space must be even. Consequently, $\tau$ must be fixed-point-free, because if $\tau(i) = i$ then this statistic above would change parity from $i-1$ to $i$, and not be always even. In particular, every stratum contains T-fixed points. This is not a common situation (e.g. it doesn't hold for $SO(2)\backslash SL(2)/B$).

**Proposition 4.3.** (1) *The map from* $\mathrm{Sp}(n)\backslash \mathrm{GL}(2n)/B$ *to fixed-point-free involutions is bijective.*

(2) *In particular,* $S_{2n}$ *gives an overcomplete system of orbit representatives. To cut it down, consider only* $\pi$ *such that* $\pi(2i), \pi(2i+1) > \pi(2i-1)$ *for all* $i$.

(3) *The* $\mathrm{Sp}(n)$*-stabilizers are connected.*

*Proof.* (1) Let $\rho$ be a fixed-point-free involution, and $\pi$ the representative described with $\pi^{-1} w_0 \pi = \rho$. ...

(2) We can permute the fixed points with $(N(T) \cap \mathrm{Sp}(n))/T$, which is $S_n \ltimes Z_2^n$. The $Z_2^n$ lets us ensure that $\pi(2i) > \pi(2i-1)$. Then the $S_n$ lets us ensure that $\pi(2i+1) > \pi(2i-1)$.

(3) If $n = 1$, $\mathrm{Sp}(1) = \mathrm{SL}(2)$ acting on $\mathbb{P}^1$, and the stabilizer is a Borel. For larger $n$, the stabilizer group of $(V, (F_i))$ maps to the stabilizer group of $(F_1^\perp/F_1, (F_i \cap F_1^\perp)/F_1))$, which is connected by induction. It remains to show that the kernel is connected.

Pick a basis starting with $\vec{v}_1 \in F_1$, $\vec{v}_{2n}$ such that $\langle v_1, v_{2n} \rangle = 1$, and the rest in $(\vec{v}_1, \vec{v}_{2n})^\perp$. Then the symplectic form looks like

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & J & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

where J is antisymmetric, and the kernel looks like

$$\begin{pmatrix} a & a\vec{r} & n \\ 0 & \mathbf{Id} & J\vec{r} \\ 0 & 0 & a^{-1} \end{pmatrix}$$

where $\vec{r}$ is isotropic for the form J. The space of those is connected.

□

## 4.2. **The twisted diagonal.**

4.2.1. *The abstract Weyl group.* Generally one says that the Weyl group $W$ of G depends on the choice of a torus $T \leq G$. Since any two T are conjugate, this choice doesn't matter up to isomorphism, but because the T are not uniquely conjugate (there are $|W|$ choices) the identification is only well-defined up to inner automorphism.

Things are better if we also make a choice of positive system $\Delta_+$ inside T's root system $\Delta$. Now the conjugating element is unique (up to T), and the identification between Weyl groups is canonical. When one has a family of things depending on some choices, but with unique isomorphisms between them, it suggests that there should be an alternate description without making these choices.

The choice of a torus T and then a positive root system gives a choice of a Borel subgroup. (This is a benefit of studying Lie theory through algebraic groups, rather than compact groups.) Let $\mathcal{B}$ denote the space of Borel subgroups of G, a homogeneous space. Since Borel subgroups are self-normalizing, if we pick a basepoint B we can identify this space with $G/B$.

Let $G_\Delta \leq G \times G$ denote the diagonal. The set of $G_\Delta$-orbits on $\mathcal{B}^2$ is called the **abstract Weyl group**, since

$$G\backslash\mathcal{B}^2 \cong G\backslash(G/B)^2 \cong (G\backslash G^2)/B^2 \cong B\backslash G/B \cong W.$$

It is not so easy to see a group structure on this set of orbits![2]

If $g \cdot (B, C)$ lies in the same $G_\Delta$-orbit as $(B, C)$, then $\theta \cdot g \cdot (B, C) = (\theta \cdot g) \cdot \theta \cdot (B, C)$ lies in the same $G_\Delta$-orbit as $\theta \cdot (B, C)$. So there is an action of $\theta$ on the abstract Weyl group.

**Lemma 4.4.** *The only orbit $\mathcal{O}_w := G_\Delta \cdot (B, wB)$ containing a positive-dimensional projective subvariety is the diagonal.*

*Proof.* If $w \neq 1$, then $w \cdot \alpha < 0$ for some simple root $\alpha$. Hence there is a G-equivariant surjection $\mathcal{O}_w \twoheadrightarrow \mathcal{O}_{r_\alpha}$, with fibers isomorphic to $(B \cap (w \cdot B))/(B \cap (r_\alpha \cdot B)) \cong \mathbb{A}^{\ell(w)-1}$. If $Y$ is a projective subvariety of $\mathcal{O}_w$, then its intersection with the (affine) fibers must be 0-dimensional, so its image in $\mathcal{O}_w$ will again be positive-dimensional.

This lets us reduce to the case $w = r_\alpha$. We want to show that any projective variety in $\overline{\mathcal{O}_{r_\alpha}} = \mathcal{O}_{r_\alpha} \cup \mathcal{O}_1$ must intersect $\mathcal{O}_1$.

On $G/B$, we have the nef line bundle $\mathcal{L}_\omega$ whose sections give the corresponding fundamental representation. Consider the restriction of $\mathcal{L}_\omega \boxtimes \mathcal{L}_{-r_\alpha \omega}$ from $(G/B)^2$ to $\overline{\mathcal{O}_{r_\alpha}}$. This turns out to have a $G_\Delta$-invariant section vanishing on $\mathcal{O}_1$.

...

□

4.2.2. *The twisted diagonal and Springer's map.* Given $(G, \theta)$, define the twisted embedding $\mathcal{B} \hookrightarrow \mathcal{B}^2$ by $B \mapsto (B, \theta \cdot B)$, and call its image $\mathcal{B}_\theta$ the **twisted diagonal**. Plainly it is invariant under the twisted diagonal subgroup $G_{\theta\Delta} := \{(g, \theta \cdot g)\}$, whose intersection with the usual diagonal $G_\Delta$ is $K_\Delta$.

Hence there is a well-defined map, defined by Springer,

$$\varphi : K_\Delta \backslash \mathcal{B}_\theta \to G_\Delta \backslash \mathcal{B}^2 \cong W$$

from $K \backslash G/B$ to the abstract Weyl group.

*Examples.*

(1) Let $G = GL(2)$, $\theta(M) = M^{-T}$, so $K = O(2)$. Then $\mathcal{B} \cong \mathbb{P}^1$, and there are two K-orbits, $\{\pm i\}$ and its complement.

(2) Let $G = GL(n)$. There is a unique G-equivariant isomorphism $\mathcal{B} \cong \mathtt{Flags}(\mathbb{C}^n)$, taking a flag to its (Borel) stabilizer. Springer's map then compares $F$ to $\theta \cdot F$. If $\theta$ is inner, conjugation by some $M \in GL(n)$, then $\theta \cdot F = M \cdot F$. Otherwise $\theta$ involves the Dynkin diagram automorphism, i.e. $\theta(g) = J^{-1} g^{-T} J$ for some $J$. Then

$$\theta \cdot (F_1 < \ldots < F_i < \ldots < F_{n-1}) = J \cdot (F_{n-1}^\perp < \ldots < F_{n+1-i}^\perp < \ldots < F_1^\perp)$$

where $\perp$ is with respect to the form with which one defines transpose.

(3) Let $G = K \times K$, with $\theta(a, b) = (b, a)$. So $\mathcal{B}(G) = \mathcal{B}(K)^2$, and the twisted diagonal inside $\mathcal{B}(G)^2$ is $\{(B, C, C, B)\} \subseteq \mathcal{B}(K)^4$. The $G_\Delta$-orbit of that is determined by the K-orbit of the first and third, and the second and fourth, i.e. $(B, C)$ and $(C, B)$. So the map

$$K \backslash G/B \to W_G$$

can be identified with $W_K \to W_G$, $w \mapsto (w, w^{-1})$.

[2] What is actually easier is to see the Demazure product. Given two orbit closures $X, Y$ in $\mathcal{B}^2$, consider the corresponding orbit closures $\overline{BwB}$, $\overline{BvB}$ in G, and multiply them in G to get something irreducible, closed, and $B \times B$-invariant.

Define a **twisted involution** in $W$ as one such that $\theta \cdot w = w^{-1}$. One way to think of it is that $\theta w$ is an involution in the semidirect product $\widetilde{W} := Z_2 \ltimes W = W \coprod \theta W$.

An alternate way to think about $\varphi$ is the following. Start with the disconnected group $\widetilde{G} := Z_2 \ltimes G$, and the map $K\backslash G \to \theta G \leq \widetilde{G}$ given by $g \mapsto g^{-1}\theta g$. This descends to

$$K\backslash G/B \to (\theta G)/(\mathrm{Ad}\, B) \twoheadrightarrow B\backslash(\theta G)/B = \theta\, (\theta \cdot B)\backslash G/B \cong \theta W$$

where the last uses the identification with the abstract Weyl group.

**Lemma 4.5.** $\varphi$ *takes values in the twisted involutions.*

*Proof.* The map takes

$$KgB \mapsto Bg^{-1}\theta gB = B\theta wB$$

for some $w \in W_G$. In the $B\backslash\theta G/B$ picture of the $\theta W$ coset of the abstract Weyl group, inversion in $\widetilde{G}$ gives the inversion in $\theta W$, and $Bg^{-1}\theta gB$ is equal to its inverse. □

For a very small example, let $G = SL(2)$, $K = S(GL(1) \times GL(1)) \cong GL(1)$, acting on $\mathbb{P}^1$ with three orbits $\{0\}$, $\{\infty\}$, and the complement.

For another family of examples, let $G = GL_n$, $K = O_n$. Then given a flag $(V_0 < \ldots < V_n) \in GL_n/B$, look at the symmetric matrix

$$m_{ij} = \dim(V_i \cap V_j^{\perp})$$

where $\perp$ is defined using the $O(n)$-invariant inner product. This $M$ turns out to necessarily be of the form $S^T\pi S$, where $\pi$ is a symmetric permutation matrix and $S$ is the upper triangular matrix of all 1s. Plainly $\pi$ is an invariant of the orbit, and it turns out that the orbits exactly correspond to involutive permutations.

### 4.3. **Finiteness of the set of orbits.**

**Lemma 4.6.** $K_\Delta$ *acts locally transitively on each* $\mathcal{B}_\theta \cap (G \cdot (B, wB))$.

*Proof.* Let $p = (B, \theta \cdot B) \in V := \mathcal{B}_\theta \cap (G \cdot (B, wB))$, acted on by $G_{\theta\Delta} \cap G_\Delta = K_\Delta$. Then

$$T_p(K_\Delta \cdot p) \leq T_p(\mathcal{B}_\theta \cap (G \cdot (B, wB))) = T_p\mathcal{B}_\theta \cap T_p(G \cdot (B, wB)),$$

and we need to prove the opposite inclusion.

$$\begin{aligned} T_{(B,\theta\cdot B)}\mathcal{B}^2 &\cong \mathfrak{g}/\mathfrak{b} \oplus \mathfrak{g}/(\theta \cdot \mathfrak{b}) \\ T_{(B,\theta\cdot B)}(G \cdot (B, \theta \cdot B)) &\cong \{(X + \mathfrak{b}, X + \theta \cdot \mathfrak{b}) : X \in \mathfrak{g}\} \\ T_{(B,\theta\cdot B)}\mathcal{B}_\theta = T_{(B,\theta\cdot B)}\,(G_{\theta\Delta} \cdot (B, \theta \cdot B)) &\cong \{(X + \mathfrak{b}, \theta \cdot X + \theta \cdot \mathfrak{b} = \theta \cdot (X + \mathfrak{b}) : X \in \mathfrak{g}\} \\ T_{(B,\theta\cdot B)}V &\cong \{(X + \mathfrak{b}, Y + \theta \cdot \mathfrak{b}) : X - Y,\ X - \theta \cdot Y \in \mathfrak{b}\} \end{aligned}$$

In particular, $2X - (Y + \theta \cdot Y) \in \mathfrak{b}$, and $Y + \theta \cdot Y \in \mathfrak{k}$, so $X \in \frac{1}{2}(Y + \theta \cdot Y) + \mathfrak{b}$.

$$T_{(B,\theta\cdot B)}K_\Delta \cdot (B, \theta \cdot B) \cong \{(Z + \mathfrak{b}, Z + \theta \cdot \mathfrak{b}) : Z \in \mathfrak{k}\}$$

To see the obvious inclusion $\supseteq$ of the last two spaces, given $Z \in \mathfrak{k}$ take $X = Y = \theta \cdot Y = Z$. For the opposite inclusion, given $X, Y$, let $Z = \frac{1}{2}(Y + \theta \cdot Y) \in \mathfrak{k}$. Then we need $Z + \mathfrak{b} = X + \mathfrak{b}$, as already shown, and we need $Z + \theta \cdot \mathfrak{b} = Y + \theta \cdot \mathfrak{b}$. Applying $\theta$, the right-hand side becomes $\theta \cdot Y + \mathfrak{b} = X + \mathfrak{b} = Z + \mathfrak{b}$, the desired left-hand side. □

**Theorem 4.7.** $K\backslash G/B$ *is finite.*

*Proof.* The decomposition of the twisted diagonal induced by intersection with the $G_\Delta$-orbits on $\mathcal{B}^2$ is finite, indexed by the Weyl group. Each piece, being algebraic, has finitely many components. By lemma 4.6, $K_\Delta$ acts transitively on each component. □

4.4. **θ-stable Borel subgroups.** A subgroup $H \leq G$ is **θ-stable** if $\theta \cdot H = H$, a concept we'll pretty much only apply to tori and Borel subgroups.

**Theorem 4.8.** *Let $\mathcal{O} = K \cdot B \subseteq \mathcal{B}$. Then the following are equivalent:*

(1) $\varphi(\mathcal{O}) = \theta$,
(2) $\mathcal{O}$ *is a closed* K*-orbit, and*
(3) B *is* θ*-stable.*

*Proof.* (1) $\iff$ (3). The intersection of the twisted diagonal and the ordinary diagonal is exactly the set of θ-stable Borel subgroups.

(3) $\implies$ (2). By the local transitivity on that closed intersection, $K \cdot (B, B)$ is a closed K-orbit.

(2) $\implies$ (1). If $K_\Delta \cdot (B, \theta \cdot B)$ is a closed orbit on $\mathcal{B}_\theta$, then it is a projective subvariety of $G_\Delta \cdot (B, wB)$. *** **If $K_\Delta$ is abelian, then we're in some extremely small case. Otherwise $K/B_K$ is positive-dimensional.** *** Then lemma 4.4 says $w = 1$. □

In particular, since the minimal K-orbits are closed,

**Corollary 4.9.** *There must exist* θ*-stable Borel subgroups.*

If we pick one, we get a non-abstract Weyl group way to think about the Springer map. Recall that for any B, we have

$$K\backslash G/B \to (\mathrm{Ad}\ B) \cdot G \to (\theta \cdot B)\backslash G/B$$

but if we pick B θ-stable, then this latter is $B\backslash G/B \cong W$.

4.5. **θ-stable tori, and Cartan classes.** It's pretty obvious that K acts on the set of θ-stable tori, and we call the orbits of this action the **Cartan classes**. The main results of this section will be that there are finitely many, and each element $K\backslash G/B$ has an associated Cartan class.

Since θ is an involution on $\mathfrak{t}$ for T θ-stable, it has a $+1$ and a $-1$ eigenspace, called the **toroidal** and **split** parts of $\mathfrak{t}$. It will be interesting to study the maximally toroidal and maximally split Cartans. Of course these two dimensions aren't independent, so we'll generally measure the **split rank** of a θ-stable Cartan.

Example. Let $G = K \times K$, and S be any Cartan at all, with projections $T_1, T_2$ to the factors. Then $S \leq T_1 \times T_2$, so by its maximality they are equal. If S is θ-stable, $T_1 = T_2$. At which point the K-conjugacy of Cartans in K implies that there is a unique Cartan class, whose split rank is $\mathrm{rank}\,(G)$.

Example. **compact Cartan** ...

Example. Let $G = SL(2), \theta \cdot M = (M^T)^{-1}, K = SO(2)$. Then K is itself a θ-stable and (completely) toroidal Cartan. Whereas the diagonal matrices form a θ-stable, split Cartan.

**Lemma 4.10.** *Every Borel subalgebra $\mathfrak{b}$ contains a θ-stable Cartan $\mathfrak{t}$, and any two are $(K \cap N)$-conjugate, where $N = [B, B]$.*

*Proof.* Let $\mathfrak{c} := \mathfrak{b} \cap (\theta \cdot \mathfrak{b})$, a $\theta$-stable solvable subalgebra, and necessarily containing Cartans. Pick a Cartan subalgebra $\mathfrak{s} \leq \mathfrak{c}$, so $\theta \cdot \mathfrak{s} \leq \mathfrak{c}$ too. By lemma 1.4, $\exists \xi \in \mathfrak{c}' = \mathfrak{n} \cap (\theta \cdot \mathfrak{n})$ with $\exp(\xi) \cdot \mathfrak{s} = \theta \cdot \mathfrak{s}$.

What would be nice is if $\mathfrak{s} = \theta \cdot \mathfrak{s}$ already. That's not usually true; we need a torus halfway between the two. So let

$$\mathfrak{s}' := \exp(\xi/2) \cdot \mathfrak{s}$$

and check that

$$\begin{aligned} \theta \cdot \mathfrak{s}' &= \theta \cdot (\exp(\xi/2) \cdot \mathfrak{s}) \\ &= (\theta \cdot \exp(\xi/2)) \cdot (\theta \cdot \mathfrak{s}) \\ &= (\theta \cdot \exp(\xi/2)) \cdot (\exp(\xi) \cdot \mathfrak{s}) \end{aligned}$$

which will be $\exp(\xi/2) \cdot \mathfrak{s}$ again if $\theta \cdot \xi = -\xi$. To see that it is, apply $\theta$ to

$$\theta \cdot \mathfrak{s} = \exp(\xi) \cdot \mathfrak{s}$$

to get

$$\begin{aligned} \mathfrak{s} &= \exp(\theta \cdot \xi) \cdot (\theta \cdot \mathfrak{s}) \\ &= \exp(\theta \cdot \xi) \cdot (\exp(\xi) \cdot \mathfrak{s}) \end{aligned}$$

hence $\exp(-\theta \cdot \xi) \cdot \mathfrak{s} = \exp(\xi) \cdot \mathfrak{s}$. By the uniqueness in lemma 1.4, $\exp(-\theta \cdot \xi) = \exp(\xi)$, and by lemma 1.3 $-\theta \cdot \xi = \xi$.

So far we have proven that this halfway-between torus $\mathfrak{s}'$ is indeed $\theta$-stable.

Now let $\mathfrak{s}_1, \mathfrak{s}_2$ be two $\theta$-stable Cartans in $\mathfrak{b}$. As before, $\exists n \in N \cap (\theta \cdot N)$ conjugating $\mathfrak{s}_1$ to $\mathfrak{s}_2$. Applying $\theta$ and using the $\theta$-stability, we learn $\theta \cdot n$ *also* conjugates $\mathfrak{s}_1$ to $\mathfrak{s}_2$, so reasoning as before $\theta \cdot n = n$. Hence $n \in N \cap K$. □

**Corollary 4.11.** *There exist $\theta$-stable tori.*

*There is a well-defined, surjective map from $K\backslash G/B$ onto the set of Cartan classes. In particular this set is finite.*

*Proof.* Since every Cartan is in a Borel, this map is surjective. This map is obviously K-invariant, so really a map from $K\backslash G/B$, which is finite. □

**Proposition 4.12.** (1) $K$ *contains regular elements of* $G$.
(2) *The closed* $K$*-orbits on* $G/B$ *correspond to* $W_G^\theta/W_K$. *If* $\theta$ *is inner, so we can pick a compact Cartan* $T_G = T_K$, *this is* $W_G/W_K$.

*Proof.* (1) Let $B \geq T$ be a $\theta$-stable Borel and torus. Then $\theta$ acts on $T$ and on the Weyl alcove, necessarily preserving the centroid of the alcove, in the interior. The exponential of that is then a regular element of $T^\theta$.
(2) Consider the action of this regular element of $T \cap K$ on $\mathcal{B}^2$. Being regular, its fixed points are all of the form $(w \cdot B, v \cdot B)$. Now we want them inside $\mathcal{B}_\Delta \cap \mathcal{B}_{\theta\Delta}$, so of the form $(w \cdot B, w \cdot B)$, where $w \cdot B$ is $\theta$-stable, i.e.

$$w \cdot B = \theta \cdot (w \cdot B) = (\theta \cdot w) \cdot (\theta \cdot B) = (\theta \cdot w) \cdot B$$

so $w \in W_G^\theta$. In particular, this group acts simply transitively on the set of $T$-fixed points on $\mathcal{B}_\Delta \cap \mathcal{B}_{\theta\Delta}$.

However, each closed $K$-orbit has an action of $W_K$ on its fixed points, so the closed $K_\Delta$-orbits correspond only to $W_G^\theta/W_K$.

□

4.5.1. *Examples.*

(1) $G = K \times K$. If $B \leq G$ is a Borel, then as with the tori $B \leq \pi_1(B) \times \pi_2(B)$ hence by maximality $B = \pi_1(B) \times \pi_2(B)$, where each projection $\pi_i(B)$ is a Borel of $K$. Then $\pi_1(B) \cap \pi_2(B)$ contains some torus $T \leq K$, and $T \times T$ is a $\theta$-stable torus in $G$.

(2) $G = GL(n), K = O(n)$ or $Sp(n/2)$. Let a **basis/scaling** mean a choice of basis of $\mathbb{C}^n$ up to scaling, i.e. a decomposition of $\mathbb{C}^n$ as a sum of a list of 1-dimensional spaces. A torus $T$ in $GL(n)$ is a choice of basis/scaling, further up to reordering, with $\theta \cdot T$ coming from the dual basis/scaling with respect to the bilinear form $\langle , \rangle$ preserved by $K$. For $T$ to be $\theta$-stable, the dual basis/scaling should be a reordering of the original, i.e. $\theta$ induces an order 2 permutation on the basis/scaling.

This has orbits of size 1 or 2, and $\langle , \rangle$ is nondegenerate on the corresponding 1- or 2-dimensional spaces. In particular, if $K = Sp(n/2)$, there are no orbits of size 1. The split rank of $T$ is given by the number of 2-cycles.

A Borel $B$ is equivalent to a choice of flag $F$ with $Stab(F) = B$. The invariant $\varphi(B)$ of the K-orbit is determined by the dimensions of the spaces $F_i \cap F_j^\perp$. (And in fact this is a complete invariant of the K-orbit for $K = O(n)$ or $Sp(n/2)$, though not for $SO(n)$.)

Lemma 4.10 then says that for each $F$, there exists a basis/scaling that is closed under $\perp$, where $F_i$ is the sum of the first $i$ lines in the basis/scaling. Can you construct it directly? (It's not quite unique, of course.)

(3) $G = GL(p+q), K = GL(p) \times GL(q)$, the fixed points of conjugation by an involution $J$. Again, a $\theta$-stable torus is given by a basis/scaling on which $J$ induces a permutation. On each of the $r$ resulting 2-d spaces, $J$ acts with eigenvalues $+1, -1$, so $r \in [0, \min(p,q)]$, and these index the Cartan classes.

As described before, the K-orbits are labeled by two sequences $(p_i), (q_i), i = 0, \ldots, n$ that each increase by $0, 1$ at each step, with $p_i + q_i \leq i$ and $p_n = p$, $q_n = q$. Then $r$ is the number of $i$ for which neither increase (and therefore also the number of $i$ for which both do).

## 5. The Bott-Samelson crank on K-orbits

Given a subset $X \subseteq G/B$, and a simple root $\alpha$, define $m_\alpha \cdot X$ as $\pi_\alpha^{-1}(\pi_\alpha(X))$, where $\pi_\alpha : G/B \to G/P_\alpha$ is the projection. These operators $m_\alpha$ give a monoid action on $K\backslash G/B$, where the $\{m_\alpha\}$ satisfy the usual commutation and braid relations, but now each $m_\alpha$ is idempotent. For any $X \subsetneq G/B$, there will be some $\alpha$ for which $m_\alpha \cdot X \supsetneq X$.

Given $X \subseteq G/B$ and a word $Q$ in the simple roots, define the **Bott-Samelson space** as

$$BS^Q X := \widetilde{X} \times^B P_{q_{|Q|}} \times^B \cdots \times^B P_{q_1}/B$$

where $\widetilde{X}$ is the preimage of $X$ in $G$. This has a natural proper map to $m_Q \cdot X$.

If $X$ is irreducible, or invariant under some subgroup $H$ of $G$, then so too will $m_\alpha \cdot X$ and any $BS^Q X$ be. In particular, if $H$ has finitely many orbits, then this monoid acts on the set of orbit closures. We assume this hereafter, especially for $H$ being $N$ or $K$.

**Lemma 5.1.** *Let $H \leq G$ be a group acting on $G/B$ with finitely many orbits, and $X$ be an irreducible $H$-orbit closure (i.e. if $H$ is connected). Let $Q$ be a word in the simple roots, and $H/S$ the unique open $H$-orbit in $m_Q \cdot X$.*

*If $H$ is simply connected and $S$ is connected, then the general fibers of $BS^Q X \to m_Q \cdot X$ are connected. If $m_Q \cdot X$ is normal, then* all *the fibers are connected. (We will most often use this in the contrapositive.)*

*If the increasing sequence $(m_{q_1 \cdots q_i} \cdot X)_{i=0,\ldots,|Q|}$ of spaces is* strictly *increasing, then the fibers are 0-dimensional. (Given any $Q$, we can drop the $q_i$ along the way for which the space doesn't increase.)*

*In particular, if $X$ is smooth (e.g. a closed $H$-orbit) and both conditions hold, then $BS^Q X \to m_Q \cdot X$ is a resolution of singularities.*

*Proof.* Since $X$ is irreducible, so is $m_Q \cdot X$, and $H/S$.

Since $BS^Q X \to m_Q \cdot X$ is $H$-equivariant, it is a bundle over $H/S$. For connectedness questions, we can replace that bundle by its Stein factorization, a covering space. Since $BS^Q X$ is irreducible, so is the preimage of $H/S$, and its Stein factorization. Hence this covering space is irreducible.

Now we need the fundamental group of $H/S$, determined from the long exact sequence $\ldots \to \pi_1(H) \to \pi_1(H/S) \to \pi_0(S) \to \ldots$ By our assumptions, this middle group vanishes, so the covering space is the trivial 1-sheeted cover, and undoing the Stein factorization we learn that fibers over $H/S$ are connected.

Zariski's Main Theorem says that birational projective morphisms to normal varieties have connected fibers.

If the increasing sequence of spaces is strictly increasing, then $\dim(m_Q \cdot X) = |Q| + \dim X = \dim(BS^Q X)$. Since an open set in $BS^Q X$ is a bundle over the open set $H/S$ in $m_Q \cdot X$, the fibers of this bundle must have dimension 0. ☐

This is most familiar in the case $H = N$, which has only connected algebraic subgroups (as it has no elements of finite order), and only one closed orbit, a point.

5.1. **Example:** $G = SL(3), K = SO(3)$**.** Recall that the orbit closures are the whole space $(0 < L < P < \mathbb{C}^3)$, $\{\mathrm{rank}\ L = 0\}$, $\{\mathrm{rank}\ P = 1\}$, and the closed orbit $\{P = L^\perp\}$.

Since a rank 2 plane $P$ contains two isotropic lines, the map $BS^1\{\mathrm{rank}\ L = 0\} \twoheadrightarrow m_1 \cdot \{\mathrm{rank}\ L = 0\} = SL(3)/B$ is generically $2:1$. (It is $1:1$ over the subvariety $\{\mathrm{rank}\ P = 1\}$.) Whereas $m_2 \cdot \{\mathrm{rank}\ L = 0\} = \{\mathrm{rank}\ L = 0\}$.

5.2. **Example:** $G = GL(2n)$, $K = Sp(n)$**.** Using the T-fixed representatives from §4.1.2, it becomes easy to compute the action of $m_\alpha$ operators. Let $\mathcal{O} = K \cdot \pi B/B \ni \pi B/B$. Then $m_\alpha \cdot \pi B/B$ is the $\mathbb{P}^1$ connecting $\pi$, $\pi r_\alpha$, whose associated involutions are $\pi^{-1} w_0 \pi$ and $r_\alpha \pi^{-1} w_0 \pi r_\alpha$.

**Proposition 5.2.** *Let $\rho$ be a fixed-point-free involution $\pi^{-1} w_0 \pi$, let $\mathcal{O}_\rho = \overline{Sp(n)\pi B}/B$ its orbit closure, and $\alpha = x_i - x_{i+1}$.*

1. *If $\rho(i) < \rho(i+1)$, then $m_\alpha \cdot \mathcal{O}_\rho = \mathcal{O}_{r_\alpha \cdot \rho}$, and the map $BS^\alpha \cdot \mathcal{O}_\rho \twoheadrightarrow \mathcal{O}_{r_\alpha \cdot \rho}$ is birational.*
2. *If $\rho(i) > \rho(i+1)$, then $m_\alpha \cdot \mathcal{O}_\rho = \mathcal{O}_\rho$. If $\rho(i) = i+1$, then $\alpha$ is an imaginary root for $\rho$, and $\rho$ and $r_\alpha$ commute. Otherwise, $\alpha$ is a complex root for $\rho$.*

*By induction,* $\dim \mathcal{O}_\rho = \dim \mathcal{O}_{\rho_{min}} + \frac{1}{2}(\ell(\rho) - \ell(\rho_{min})) = n^2 + \frac{1}{2}(\ell(\rho) - n)$.

*Proof.* (1) Always we have $m_\alpha \cdot \rho \ni r_\alpha \cdot \rho$, so $m_\alpha \cdot \mathcal{O}_\rho \supseteq \mathcal{O}_{\rho r_\alpha}$. By the assumption, $\dim \mathcal{O}_{\rho r_\alpha} = 1 + \dim \mathcal{O}_\rho$, which is an upper bound on $\dim m_\alpha \cdot \mathcal{O}_\rho$, so the dimensions match. Since $m_\alpha \cdot \mathcal{O}_\rho$ is irreducible, the inclusion is an equality.

By proposition 4.3 (3), we can apply the first two conclusions of lemma 5.1 to the map $BS^\alpha \mathcal{O}_\rho \to \mathcal{O}_{r_\alpha}$, and learn it is birational.

(2)

(3)

□

5.3. **Types of roots and the monoid action.** We want to figure out when the map $BS^\alpha \nu \to m_\alpha \cdot \nu$ is generically 1:1, 2:1, or $\mathbb{P}^1$:1. Also, how many different $\nu$ have $m_\alpha \cdot \nu = \nu'$. Instead of saying "generically", we can take the open K-orbit in $m_\alpha \cdot \nu$, and decompose its preimage in $BS^\alpha \nu$ into K-orbits.

**Lemma 5.3.** *If two points* $e, f \in \pi_\alpha^{-1}(gP_\alpha/P_\alpha)$ *are conjugate by some* $h \in G$*, then* $h \in g \cdot P_\alpha$.

*Proof.* Since $g \cdot P_\alpha$ acts transitively on $\pi_\alpha^{-1}(gP_\alpha/P_\alpha)$, it is enough to consider $e = f$, and one can further reduce to $g = 1$, $e = f = P_\alpha$. Then the statement is that $P_\alpha$ is self-normalizing. □

Let $gB/B$ be in the open K-orbit in $\nu$, and $F = \pi_\alpha^{-1}(gP_\alpha/P_\alpha)$, so $K \cdot F$ is open in $m_\alpha \cdot \nu$. Then by lemma 5.3, the K-orbits on $K \cdot F$ correspond to the $K \cap (g \cdot P_\alpha)$-orbits on F. Let $S \leq Aut(F)$ be the image of this group in $Aut(F) \cong PSL(2)$. There are not many possibilities for this group, and its (finite) orbit structure on F!

Fix a $\theta$-stable torus T with root system $\Delta$. Recall that a root $\beta \in \Delta$ is called

- **real** if $\theta \cdot \beta = -\beta$.
- **imaginary** if $\theta \cdot \beta = \beta$. In this case, $\theta$ acts on $\mathfrak{g}_\beta$. Call the root
  - **compact imaginary** if this action is trivial,
  - **noncompact imaginary** if it is by $-1$
- **complex** otherwise.

In the real and imaginary cases, $\theta$ acts on the corresponding $SL_2$ (or $PSL_2$) subgroup. In the compact imaginary case, $\theta$ acts trivially. In the real and noncompact imaginary cases, the fixed points (i.e. the intersection with K) are either $SO(2)$ or $O(2)$.

In the complex case, the roots $\alpha$ and $\theta \cdot \alpha$ have the same length, so the rank 2 subsystem generated by them is either $A_1 \times A_1$ or $A_2$. It has a nontrivial action of $\theta$, with fixed points either the diagonal $A_1$ or the $SO(3) \leq SL(3)$.

Let us now assume also that B is $\theta$-stable (which eliminates the possibility of real roots). The intersection just described of K with the rank 2 subgroup is an $SL_2$, and then with $P_\alpha$ is a Borel of that $SL_2$.

**Theorem 5.4.** *Let* $(B, T)$ *be* $\theta$*-stable,* $\nu = \overline{K \cdot gB}/B$, F *the fiber over* $gB/B$ *of* $\pi_\alpha : G/B \to G/P_\alpha$, *and* $S \leq Aut(F)$ *the image of* $K \cap (g \cdot P_\alpha)$.

(1) *If* $\alpha$ *is compact imaginary for* $\varphi(\nu)$, *then* $S = Aut(F)$, *with one orbit, and* $m_\alpha \cdot \nu = \nu$. *That is, the map* $BS^\alpha \nu \twoheadrightarrow m_\alpha \cdot \nu$ *is everywhere* $\mathbb{P}^1 : 1$.

(2) *If $\alpha$ is complex for $\varphi(\nu)$, then $S \neq \mathrm{Aut}(F)$ is a Borel subgroup of $\mathrm{Aut}(F)$, with one $\mathbb{A}^1$-orbit and one fixed point (corresponding to $m_\alpha \cdot \nu, \nu$ respectively). The map $BS^\alpha \nu \twoheadrightarrow m_\alpha \cdot \nu$ is generically $1:1$, and no other $\nu'$ has $m_\alpha \cdot \nu' = m_\alpha \cdot \nu$.*
(3) *If $\alpha$ is noncompact imaginary or real for $\varphi(\nu)$, then $S$ is conjugate to either $SO(2)$ or $O(2)$. In the $SO(2)$ case, $S$ has three orbits, and the two point orbits correspond to two $K$-orbit closures $\nu, \nu'$ with $m_\alpha \cdot \nu' = m_\alpha \cdot \nu$. Each of $BS^\alpha\nu, BS^\alpha\nu'$ map birationally to $m_\alpha \cdot \nu$.*
   *In the $O(2)$ case, there is only one such $K$-orbit $\nu$, and its map $BS^\alpha\nu \twoheadrightarrow m_\alpha \cdot \nu$ is generically $2:1$.*

5.4. **Example:** $G = GL(n)$, $K = O(n)$ **or** $Sp(n/2)$**.** Let $J$ be antidiagonal and $\theta(M) = J^{-1}M^{-T}J$, so that the usual $(B,T)$ are $\theta$-stable. Then the action of $\theta$ on $\Delta_+$ is by $-w_0$. Then as in §4.1.2, the orbits are classified by involutions $\pi$ with $\dim(F_i \cap F_{2n-j}^\perp) = \#1$s in the SW $i \times j$ of $\pi$, but now $\pi$ can have fixed points.

For an orbit $\nu$ with associated involution $\pi \in S_n$ (necessarily fixed-point-free, if $J$ is antisymmetric), the action of $\varphi(\nu)$ on $\Delta$ is by $-\pi$.

Hence $x_i - x_{i+1}$ is

- real for $\nu$ if $\pi$ fixes $i, i+1$ (impossible if $K = Sp(n/2)$)
- imaginary for $\nu$ if $\pi$ swaps $i, i+1$ (always compact for $K = Sp(n/2)$, and always noncompact for $K = O(n)$)
- complex for $\nu$ otherwise.

Compare this with `http://atlas.math.umd.edu/web/atlasInput.html`, taking the group to be "GL(n, R)", or $n$ even and the group to be "GL(n, H)". Each row consists of

- The orbit number (an arbitrary numbering)
- The "length" := dimension minus dimension of a closed orbit
- The Cartan class of the orbit (in an arbitrary numbering of the classes)
- The types of the simple roots: **C**omplex, **r**eal, **c**ompact/**n**oncompact imaginary
- Two lists, describing an action of the simple reflections on the orbits, which we haven't fully defined yet
- A reduced expression for $\varphi(\nu)$ as an element of $W$ not $\theta W$.

Orbit #0 at the top is the closed orbit, corresponding to the involution $w_0$, and the bottom orbit is the open one corresponding to the involution $e$ for $K = O(n)$ or $(2i \leftrightarrow 2i-1)_{i \leq n/2}$ for $K = Sp(n/2)$. The Cartan class is determined by the number of fixed points (so, constant in the $K = Sp(n/2)$ case).

## 6. The weak and strong Bruhat orders on $K\backslash G/B$

The **strong Bruhat order** on $K\backslash G/B$ is given by inclusion of orbit closures. The **weak Bruhat order** is the transitive extension of the Richardson-Springer monoid action. It is easy to see that $\nu_1 \leq \nu_2$ in weak order implies the same in strong order.

## 7. $\mathcal{D}$-modules and Harish-Chandra $(\mathfrak{g}, K)$-modules

## 8. Branching from G to K

In this section $K$ is connected, and $\lambda, \mu$ denote dominant weights for $G, K$ respectively. The dominant weights $(T_G^*)_+$ of a connected group correspond to two important sets: the irreducible representations, and the line bundles over the flag manifold that have sections. By homogeneity, the sections automatically generate the line bundles (there's no place at which all the sections vanish), and are called **nef** for "numerically effective", which is weaker than "ample".

Given a closed K-orbit $\nu_{\text{closed}} \subseteq G/B$, we get a natural map $\rho_{\nu_{\text{closed}}} : (T^*_G)_+ \to (T^*_K)_+$ by restricting a nef line bundle from $G/B$ to $\nu_{\text{closed}}$.

Recall that if we fix a $\theta$-stable Borel $B \leq G$, the closed K-orbits correspond to the minimal-length coset representatives for $W^\theta_G/W_K$. Here B itself lies in the closed K-orbit $\nu_B$ corresponding to the identity in $W^\theta_G$.

We generalize the problem of decomposing

$$V^G_\lambda = \Gamma(G/B; \mathcal{L}_\lambda)$$

as a K-representation, to the more general problem of decomposing $\Gamma(\nu; \mathcal{L}_\lambda)$, where $\nu$ is a K-orbit closure that will be fixed for the rest of this section, and $\mathcal{L}_\lambda$ is the Borel-Weil line bundle that has been silently restricted from $G/B$.

The main trick is to use a $\xi \in T^*_+$ such that $\Gamma(\nu; \mathcal{L}_\xi)$ contains a section $\xi$ that is K-invariant, not zero, but does vanish *somewhere* (rather than trivializing the restricted bundle). That is impossible if $\nu$ is a K-orbit (i.e. closed), but luckily in that case the problem is already solved by Borel-Weil for K:

$$\Gamma(\nu; \mathcal{L}_\lambda) = V^K_{\rho_\nu(\lambda)}.$$

**Proposition 8.1.** *Let $\lambda \in (T^*_G)_+$, and $\widehat{\lambda} = (1 - \varphi(\nu)) \cdot \lambda$, where $\varphi(\nu)$ is the involution in $\theta W$ corresponding to the orbit $\nu$.*

1. *The line bundle $\mathcal{L}_{\widehat{\lambda}}$ on $\nu$ has a nonvanishing K-invariant section $\Psi_\nu$.*
2. *If $\lambda$ is regular dominant, $\Psi_\nu$ vanishes on all the smaller K-orbits.*
3. *For some fundamental weight, $\Psi_\nu$ vanishes on some smaller K-orbit.*

*Proof.* Let $\widetilde{\nu} \subseteq G$ be the preimage of $\nu \subseteq G/B$. Then thinking of $\lambda$ as $B \to B/B' \cong T \to \mathbb{C}^\times$,

$$\Gamma(\nu; \mathcal{L}_\lambda) \quad \cong \quad \{\widetilde{\Psi} : \widetilde{\nu} \to \mathbb{C} \mid \widetilde{\Psi}(gb^{-1}) = b^\lambda f(g)\}.$$

(The $b \mapsto b^{-1}$ isn't really important when calculating with line bundles, since $GL_1$ is abelian, but it is for vector bundles.)

Since we want $\widetilde{\Psi}$ to satisfy $\widetilde{\Psi}(kg) = \widetilde{\Psi}(g)$, it suffices for it to be a function of $\theta(g)^{-1}g$. Recall that $\{\theta(g)^{-1}g\} \subseteq G$ lies inside a unique smallest $\overline{BwB}$, and that $\varphi(\nu) = \theta w$.

Let $m : G \to \mathbb{C}$ be the function

$$m(g) = \langle w \cdot \vec{v}_{w_0 \cdot \lambda}, g \cdot \vec{v}_{-\lambda} \rangle$$

where $\vec{v}_{w_0 \cdot \lambda}, \vec{v}_{-\lambda}$ are the low weight vectors of the G-irreps corresponding to $\lambda$ and its dual. This function is obviously a right B-weight vector with weight $-\lambda$, and *when restricted to* $\overline{BwB}$ it is a left B-weight vector with weight $-w \cdot \lambda$. If $\lambda$ is regular dominant, then $m$ vanishes on all smaller $B \times B$-orbit closures.

Finally, define $\widetilde{\Psi}(g) = m(\theta(g)^{-1}g)$. Then

$$\widetilde{\Psi}(gb^{-1}) = m(\theta(b)\theta(g)^{-1}gb^{-1}) = \theta(b)^{-w \cdot \lambda} b^\lambda m(\theta(g)^{-1}g) = b^{\lambda - \varphi(\nu) \cdot \lambda} \widetilde{\Psi}(g)$$

If $\lambda = \sum_i n_i \omega_i$, where the $\omega_i$ are the fundamental weights, then $\widetilde{\Psi}_\lambda = \prod_i (\widetilde{\Psi}_{\omega_i})^{n_i}$, so none of the ones with $n_i > 0$ can be zero, and at least one of them must vanish somewhere.

□

Pick $\xi_\nu \in T^*_G$ such that $\mathcal{L}_{\xi_\nu}$ has a K-invariant section $\Psi_\nu$ that, although not zero, does vanish somewhere. (For $\nu$ not closed, the above proposition says that some $(1 - \varphi(\nu)) \cdot \omega$ will work.) Then we have a short exact sequence of sheaves (exactness proved below)

$$0 \to \mathcal{L}_{-\xi_\nu} \to \mathcal{O}_\nu \to \mathcal{O}_{\Psi_\nu = 0} \to 0$$

which we can tensor with $\mathcal{L}_\lambda$ and get the crucial long exact sequence:

$$0 \to \Gamma(\nu; \mathcal{L}_{\lambda - \xi_\nu}) \to \Gamma(\nu; \mathcal{L}_\lambda) \to \Gamma\big(\{\Psi = 0\}; \mathcal{L}_\lambda\big) \to \cdots$$

**Proposition 8.2.**

### 8.1. **Asymptotic representation theory.**

### 8.2. $(G, K)$**-paths.**

### 8.3. **An asymptotic branching rule.**

## 9. The Matsuki correspondence: proof

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