Let
\[ M = \begin{pmatrix} 1 & 0 & 5 & -3 & 2 & 8 & 0 \\ 4 & 4 & 0 & 3 & 0 & -3 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ 2 & 0 \\ 1 & 1 \\ 3 & 0 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}. \]

Compute \( \det(MN) \) and \( \det(NM) \).

A. \( MN \) is an easily calculated \( 2 \times 2 \) matrix whose determinant happens to be nonzero. \( NM \) is a lengthily calculated \( 7 \times 7 \) matrix that cannot be invertible, since its image has dimension at most 2. (In fact it’s exactly 2.) So its determinant is zero.

Let \( Q = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\} \) be the first quadrant, and consider the map
\[ \phi : Q \to Q, \quad (x, y) \mapsto (x + y, x, \sqrt{y}). \]

a. Compute the derivative of \( \phi \).

A.
\[ \begin{pmatrix} 1 \\ \sqrt{y} \\ x/(2\sqrt{y}) \end{pmatrix} \]

b. This isn’t a parametrization, as the derivative sometimes drops rank.

Find the equation of the curve \( C \) inside \( Q \) where this happens.

A. The determinant is zero where \( \sqrt{y} - x/2\sqrt{y} = 0 \), i.e. \( x = 2y \).

c. Find the equation of its image \( \phi(C) \).

A. If \( u = x + y = 3y \), and \( v = x\sqrt{y} = 2y\sqrt{y} \), then \( v = 2(u/3)^{3/2} = \frac{2}{3\pi}u^{3/2} \).

d. Say we wanted to Lebesgue integrate the function \( e^{-x} \) on \( Q \), in the (unbounded) region below the curve \( \phi(C) \). The change of variable formula would let us turn it into a different integral, by pulling back along \( \phi \); what would this integral be? (Don’t evaluate it.)
A. It’s easier to work in different coordinates on the two $\mathbb{R}^2$s, i.e. $(u, v) = (x + y, x\sqrt{y})$ on the target. Then the form we’re pulling back is

$$e^{-u} du \wedge dv = e^{-(x+y)} d(x+y) \wedge d(x\sqrt{y})$$

$$= e^{-(x+y)} (dx + dy) \wedge (\sqrt{y}dx + x \ dy/(2\sqrt{y}))$$

$$= e^{-(x+y)} (dx + dy) \wedge (\sqrt{y}dx + x \ dy/(2\sqrt{y}))$$

$$= e^{-(x+y)} (\sqrt{y}dx \wedge dx + x/(2\sqrt{y})dy \wedge dy)$$

$$+ \sqrt{y}dy \wedge dx + x/(2\sqrt{y})dy \wedge dy)$$

$$= e^{-(x+y)} (x/(2\sqrt{y})dx \wedge dy - \sqrt{y}dx \wedge dy)$$

$$= e^{-(x+y)} (x/(2\sqrt{y}) - \sqrt{y}) \ dx \wedge dy$$

At this point I should have had a practice question about setting up (not doing) each of these integrals via Fubini; first $x$ then $y$ or vice versa, in each coordinate system.

e. Let $\alpha = x \ dy$. Compute the pullback $\phi^*(\alpha)$.

A. Same sorta thing:

$$u \ dv = (x+y)d(x\sqrt{y}) = (x+y)(\sqrt{y}dx + x/(2\sqrt{y})dy$$

Buncha true/false about whether continuous implies integrable and whatnot. Basically the answer is always “false”, so the point is to give a counterexample.

Let $\omega = \sum_{i=1}^{n} dx_i \wedge dx_{n+i}$. Let $\nu = \omega \wedge \omega \wedge \cdots \wedge \omega$ ($n$ factors).

a. What’s $\nu$ when $n = 2$?

b. What’s $\nu$ when $n = 3$?

c. What’s $\nu$ for general $n$?

A. A priori when we wedge these $n$ factors together, each containing $n$ terms, we have $n^n$ terms to look at. But if there’s any repetition
in the factors, the wedge dies, so only the $n!$ terms with no repetition survive.
Also, those terms are all equal, because we can commute 2-forms past each other, to make them all look like

$$(dx_1 \wedge dx_{n+1}) \wedge (dx_2 \wedge dx_{n+2}) \wedge \cdots$$

so we only need to figure out the signs involved in turning that into $dx_1 \wedge \cdots \wedge dx_{2n}$.

Imagine sorting the large $dx_i$ towards the end. The $dx_{2n}$ is already there, good, but each $dx_{i+n}$ needs to move past all the $dx_j$, $j > i$, each time incurring a factor of $-1$. That’s $0 + 1 + \cdots + (n-1)$ such factors, which adds up to $\binom{n}{2}$. Then $(-1)^{\binom{n}{2}}$ is $+1$ for $n \equiv 0, 3 \pmod{4}$, and $-1$ for $n \equiv 1, 2 \pmod{4}$.

Final answer: $(-1)^{\binom{n}{2}} n! \, dx_1 \wedge \cdots \wedge dx_{2n}$.

Let $f : \mathbb{R} \to \mathbb{R}^2$ take $t \mapsto (\cos e^t, \sin e^t)$.

a. Compute the derivative.

b. Let $\alpha = b(x,y) \, dx + c(x,y) \, dy$ be a general 1-form on $\mathbb{R}^2$. Compute $f^*(\alpha)$.

A. This is just like the question two above.

Let $V, W$ be vector spaces with orientations.

a. Say how to define an orientation on $V \times W = \{(\vec{v}, \vec{w}) \in V \times W\}$.

A. To put an orientation on a vector space, it suffices to state a basis and then say “that one’s positively oriented”.

Here, the answer people usually like is to take a positive basis $(v_1, \ldots, v_a)$ of $V$, another one $(w_1, \ldots, w_b)$ of $W$, soup them up to vectors in $V \times W$, and concatenate the lists to get a basis

$$((v_1, \vec{0}), \ldots, (v_a, \vec{0}), (\vec{0}, w_1), \ldots, (\vec{0}, w_b))$$

of $V \times W$. 

b. Define

\[ \sigma : V \times W \to W \times V, \quad (\vec{v}, \vec{w}) \mapsto (\vec{w}, \vec{v}). \]

When is it orientation-preserving, when orientation-reversing?

A. We take a positive basis of \( V \times W \) (such as the above), hit it with \( \sigma \), and compare that to a positive basis of \( W \times V \).

The positive basis we would expect to see of \( W \times V \), following the recipe above, is

\[
( (w_1, \vec{0}) , \ldots , (w_b, \vec{0}) , (\vec{0}, v_1) , \ldots , (\vec{0}, v_a) ).
\]

If we hit part (a)'s basis with \( \sigma \), we get

\[
( (\vec{0}, v_1) , \ldots , (\vec{0}, v_a) , (w_1, \vec{0}) , \ldots , (w_b, \vec{0}) ).
\]

So how many switches does it take to turn one of these bases of \( W \times V \) into the other? We need to carry each of these \( a \) vectors past each of those \( b \) vectors, so \( ab \) switches.

Hence the map is orientation-preserving if \( a \) or \( b \) are even, and reversing if \( a \) and \( b \) are both odd.