Name: ____________________________

Test version: #2

Numbers in boxes are the point scores for their questions.

1a. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function (in particular, bounded with bounded support), and $h : \mathbb{R} \to \mathbb{R}$ be the Heaviside function $h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$

State in full the definition of $f$ being integrable, specifically in dimension $n = 1$ (i.e. don’t just parrot the general definition) [15].

For each $N \in \mathbb{N}$ and $k \in \mathbb{Z}$, let

$$C_{N,k} = \left[ k/2^N, (k + 1)/2^N \right)$$

and

$$L_N = 2^{-N} \sum_{k \in \mathbb{Z}} \inf_{C_{N,k}} f, \quad U_N = 2^{-N} \sum_{k \in \mathbb{Z}} \sup_{C_{N,k}} f.$$

Then $f$ is integrable iff $\lim_{N \to \infty} L_N = \lim_{N \to \infty} U_N$, or equivalently, if $\forall \epsilon > 0 \exists N$ such that $U_N - L_N < \epsilon$.

b. Prove that $f \cdot h$ is integrable, either from the definition, or from our characterization in terms of “measure zero” sets [25].

(From the definition.) Let $L'_N, U'_N$ be the corresponding sums for $f \cdot h$, and $\text{osc}_S f = \sup_S f - \inf_S f$.

$$U_N - L_N = \frac{\sum_{k \in \mathbb{Z}} \text{osc}_{C_{N,k}} f}{2^N} \geq \frac{\sum_{k \geq 0} \text{osc}_{C_{N,k}} f}{2^N} = \frac{\sum_{k \in \mathbb{Z}} \text{osc}_{C_{N,k}} (f \cdot h)}{2^N} = U'_N - L'_N.$$

So to get $U'_N - L'_N < \epsilon$, it suffices to get $U_N - L_N < \epsilon$, which we know we can do for large $N$ since $f$ was assumed integrable.

(From the characterization.) Since $f$ is integrable, its set $\Delta$ of discontinuities is measure zero. Adding $0$ to $\Delta$ keeps it measure zero. Then
\( f \cdot h \)'s set of discontinuities is contained in \( \Delta \cup \{0\} \), so is measure zero. Hence \( f \cdot h \) is integrable.

2. The 3-cube \([0, 1]^3\) breaks up into several pieces, according to the order of the coordinates, for example the two pieces

\[
\{(x, y, z) \in [0, 1]^3 : x = z > y\}, \quad \{(x, y, z) \in [0, 1]^3 : y > x > z\}.
\]

List all the pieces that have nonzero 3-volume \( 10 \) and compute the 3-volume of each piece \( 20 \). Justify your calculations of these volumes.

There are thirteen possible orders,

\( x = y = z, \ x = y > z, \ x = y < z, \ x = z > y, \ x = z < y, \ y = z > x, \ y = z < x, \ x > y > z, \ x > z > y, \ y > x > z, \ y > z > x, \ z > x > y, \ z > y > x \)

so thirteen pieces. Only the latter six have nonzero 3-volume.

By symmetry (or the change-of-variable formula, if you want to get pedantic), those six must all have the same 3-volume. Putting them together, we get \([0, 1]^3\) minus a set with zero 3-volume. Hence, each must have 3-volume \( \frac{1}{6} \).

Or, you could look at the triple integral

\[
\int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{y} 1 \, dz \, dy \, dx = \int_{x=0}^{1} \int_{y=0}^{x} y \, dy \, dx
\]

\[
= \int_{x=0}^{1} \frac{x^2}{2} \, dx
\]

\[
= \frac{x^3}{6} \bigg|_{0}^{1}
\]

\[
= \frac{1}{6}
\]
3. Let $A$ be an $d \times d$ matrix. If the limit $\lim_{n \to \infty} \det(A^n)$ exists, what might $\det A$ be? List all possibilities of $\det A$.

Prove that the answers you found are the only possibilities, and give examples of $A$ for each such answer.

Since $\det(A^n) = (\det A)^n$, we’re looking at powers of $\det(A)$.

If $|\det(A)| > 1$, then $|\det(A)|^n$ goes to $\infty$, so no limit.

If $|\det(A)| < 1$, then $|\det(A)|^n$ goes to $0$. Easiest example: $A = [0]$.

If $|\det(A)| = 1$, then $\det(A)^n$ just wanders around the unit circle, never having a limit unless $\det(A) = 1$. Easiest example: $A = [1]$.

So in all, either $|\det(A)| < 1$ or $\det(A) = 1$, and in either case the $1 \times 1$ matrix $[\det(A)]$ gives an example.