Prof. Allen Knutson's Math 2240 Midterm, February 28, 2012

Name: ________________________

Test version: #2

| Numbers in boxes | are the point scores for their questions.

1a. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function (in particular, bounded with bounded support), and $h : \mathbb{R} \to \mathbb{R}$ be the Heaviside function

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

State in full the definition of f being integrable, specifically in dimension $n = 1$ (i.e. don't just parrot the general definition) $\boxed{15}$.

For each $N \in \mathbb{N}$ and $k \in \mathbb{Z}$, let

$$C_{N,k} = [k/2^N, (k+1)/2^N)$$

and

$$L_N = 2^{-N} \sum_{k \in \mathbb{Z}} \inf_{C_{N,k}} f, \qquad U_N = 2^{-N} \sum_{k \in \mathbb{Z}} \sup_{C_{N,k}} f.$$

Then f is integrable iff $\lim_{N\to\infty} L_N = \lim_{N\to\infty} U_N$, or equivalently, if $\forall \epsilon > 0\ \exists N$ such that $U_N - L_N < \epsilon$.

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b. Prove that $f \cdot h$ is integrable, either from the definition, or from our characterization in terms of "measure zero" sets $\boxed{25}$.

(From the definition.) Let L'_N, U'_N be the corresponding sums for $f \cdot h$, and $\mathrm{osc}_S f = \sup_S f - \inf_S f$.

$$U_N - L_N = \frac{\sum_{k \in \mathbb{Z}} \mathrm{osc}_{C_{N,k}} f}{2^N} \geq \frac{\sum_{k \geq 0} \mathrm{osc}_{C_{N,k}} f}{2^N} = \frac{\sum_{k \in \mathbb{Z}} \mathrm{osc}_{C_{N,k}} (f \cdot h)}{2^N} = U'_N - L'_N.$$

So to get $U'_N - L'_N < \epsilon$, it suffices to get $U_N - L_N < \epsilon$, which we know we can do for large N since f was assumed integrable.

(From the characterization.) Since f is integrable, its set Δ of discontinuities is measure zero. Adding 0 to Δ keeps it measure zero. Then

$f \cdot h$'s set of discontinuities is contained in $\Delta \cup \{0\}$, so is measure zero. Hence $f \cdot h$ is integrable.

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2. The 3-cube $[0,1]^3$ breaks up into several pieces, according to the order of the coordinates, for example the two pieces

$$\{(x,y,z) \in [0,1]^3 : x = z > y\}, \qquad \{(x,y,z) \in [0,1]^3 : y > x > z\}.$$

List all the pieces that have nonzero 3-volume $\boxed{10}$, and compute the 3-volume of each piece $\boxed{20}$. Justify your calculations of these volumes.

There are thirteen possible orders,

$$x = y = z,\ \ x = y > z,\ \ x = y < z,\ \ x = z > y,\ \ x = z < y,\ \ y = z > x,\ \ y = z < x,$$

$$x > y > z,\ \ x > z > y,\ \ y > x > z,\ \ y > z > x,\ \ z > x > y,\ \ z > y > x$$

so thirteen pieces. Only the latter six have nonzero 3-volume.

By symmetry (or the change-of-variable formula, if you want to get pedantic), those six must all have the same 3-volume. Putting them together, we get $[0,1]^3$ minus a set with zero 3-volume. Hence, each must have 3-volume $= \frac{1}{6}$.

Or, you could look at the triple integral

$$\begin{aligned} \int_{x=0}^{1} \int_{y=0}^{x} \int_{z=0}^{y} 1 \, dz \, dy \, dx &= \int_{x=0}^{1} \int_{y=0}^{x} y \, dy \, dx \\ &= \int_{x=0}^{1} \frac{x^2}{2} \, dx \\ &= \left. \frac{x^3}{6} \right|_0^1 \\ &= \frac{1}{6} \end{aligned}$$

3. Let A be an $d \times d$ matrix. If the limit $\lim_{n\to\infty} \det(A^n)$ exists, what might $\det A$ be? List all possibilities of $\det A$ $\boxed{10}$.
Prove that the answers you found are the only possibilities, and give examples of A for each such answer $\boxed{20}$.

Since $\det(A^n) = \det(A)^n$, we're looking at powers of $\det(A)$.
If $|\det(A)| > 1$, then $|\det(A)|^n$ goes to ∞, so no limit.
If $|\det(A)| < 1$, then $|\det(A)|^n$ goes to 0. Easiest example: $A = [0]$.
If $|\det(A)| = 1$, then $\det(A)^n$ just wanders around the unit circle, never having a limit unless $\det(A) = 1$. Easiest example: $A = [1]$.
So in all, either $|\det(A)| < 1$ or $\det(A) = 1$, and in either case the 1×1 matrix $[\det(A)]$ gives an example.