Prof. Allen Knutson's Math 2240 Midterm, April 11, 2012

Numbers in boxes are the point scores for their questions.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. If f is Lebesgue integrable, is f^2 therefore Lebesgue integrable? Well, yes or no? $\boxed{10}$

Prove that f^2 is L-integrable, or give a counterexample (in which you prove f is L-integrable but f^2 is not). $\boxed{25}$

Nope.

If f were R-integrable, then so too would f^2 be. So we need to violate one or more of the requirements for R-integrability.

Let's look for a convergent sum of rectangle areas $\sum_n b_n h_n$ with $\sum b_n h_n^2 = \infty$. If $b_n h_n = \frac{1}{n^2}$, but $b_n h_n^2 = \frac{1}{n}$, we're in business. So take $h_n = n$, $b_n = \frac{1}{n^3}$, and consider the function

$$f = \sum_n 1_{[n,n+b_n)} h_n = \sum_n 1_{[n,n+\frac{1}{n^3})} n.$$

By the fact that $\sum_n \frac{1}{n^2}$ converges, this is L-integrable, but its square $\sum_n 1_{[n,n+\frac{1}{n^3})} n^2$ is not.

2a. Consider the curve along the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the saddle surface $z = x^2 - y^2$.

Find a parametrization γ of an open set in this curve. $\boxed{15}$

Let $X = x^2, Y = y^2$, so the equations become $X + Y + z^2 = 1, z = X - Y$.

For fixed x, we have

$$Y = 1 - z^2 - X = 1 - (X - Y)^2 - X = 1 - X^2 + 2XY - Y^2$$

so

$$Y^2 + (1 - 2X)Y + (X^2 - 1) = 0$$

Let Y be the positive root of this,

$$Y = \frac{2X - 1 + \sqrt{(1 - 2X)^2 - 4(X^2 - 1)}}{2} = X - \frac{1 - \sqrt{5 - 4X}}{2}$$

so

$$X - Y = \frac{1 - \sqrt{5 - 4X}}{2}.$$

Hence for each X with $0 < 4X < 5$, we get a point

$$\left(\sqrt{X}, \quad \sqrt{X - \frac{1 - \sqrt{5 - 4X}}{2}}, \quad \frac{1 - \sqrt{5 - 4X}}{2}\right)$$

in the curve.

Compute the derivative of the function you use. 10

$$\frac{dZ}{dX} = \frac{d}{dX}\frac{1 - \sqrt{5 - 4X}}{2} = \frac{2}{\sqrt{5 - 4X}}$$

So the derivative of

$$X \mapsto \left(\sqrt{X}, \quad \sqrt{X - \frac{1 - \sqrt{5 - 4X}}{2}}, \quad \frac{1 - \sqrt{5 - 4X}}{2}\right)$$

is

$$\left(\frac{-1}{2\sqrt{X}}, \quad \frac{1 - \frac{2}{\sqrt{5-4X}}}{2\sqrt{X - \frac{1-\sqrt{5-4X}}{2}}}, \quad \frac{2}{\sqrt{5 - 4X}}\right)$$

2b. Let $\alpha = x\ dx$. Write the integral of α along the image of your γ as an ordinary integral on an interval. (Don't actually solve that integral.) 20

$$\int_X \sqrt{X}\frac{-1}{2\sqrt{X}}\ dX = -\frac{1}{2}\int_X\ dX$$

3. Recall that if α is a j-form on V, and β a k-form, then $\alpha \wedge \beta$ is the $(j+k)$-form defined by

$$(\alpha \wedge \beta)\,(\vec{v}_1, \ldots, \vec{v}_{j+k})$$

$$:= \sum_{\substack{S \subseteq \{1,\ldots,j+k\} \\ |S|=j, T=\{1,\ldots,j+k\}\setminus S}} (-1)^{\#\{(s\in S, t\in T): s>t\}} \alpha(\vec{v}_{s_1}, \vec{v}_{s_2}, \ldots, \vec{v}_{s_k}) \beta(\vec{v}_{t_1}, \vec{v}_{t_2}, \ldots, \vec{v}_{t_j})$$

e.g. if $j = k = 1$, then $(\alpha \wedge \beta)(\vec{v}, \vec{w}) = \alpha(\vec{v})\beta(\vec{w}) - \alpha(\vec{w})\beta(\vec{v})$.

Recall dx_i is the 1-form that takes a vector to its ith entry.

Let ω be the 2-form on 4-space defined by

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_1 \wedge dx_3.$$

Compute $\omega \wedge \omega$, in terms of the standard basis for forms. $\boxed{20}$

By the definition,

$$\begin{aligned} & (\omega \wedge \omega)(v_1, v_2, v_3, v_4) \\ = \; & \omega(v_1, v_2)\omega(v_3, v_4) - \omega(v_1, v_3)\omega(v_2, v_4) + \omega(v_1, v_4)\omega(v_2, v_3) \\ & + \omega(v_2, v_3)\omega(v_1, v_4) - \omega(v_2, v_4)\omega(v_1, v_3) + \omega(v_3, v_4)\omega(v_1, v_2) \\ = \; & 2\omega(v_1, v_2)\omega(v_3, v_4) - 2\omega(v_1, v_3)\omega(v_2, v_4) + 2\omega(v_1, v_4)\omega(v_2, v_3) \end{aligned}$$

Since it is a 4-form on 4-space, it is a multiple of the only basis element,

$$(v_1, v_2, v_3, v_4) \mapsto \det \left[v_1 v_2 v_3 v_4 \right]$$

and to figure out which, we only have to feed in $\left[v_1 v_2 v_3 v_4 \right]$ = the identity matrix. In the above, that becomes

$$\begin{aligned} &= 2 \cdot 1 \cdot 1 - 2 \cdot 1 \cdot 0 + 2 \cdot 0 \cdot 0 \\ &= 2. \end{aligned}$$

Hence $\omega \wedge \omega = 2 dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$.