REPRESENTATIONS OF U(N) – THE BOREL-WEIL THEOREM NOTES FOR MATH 261, FALL 2001

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This is just barely the amount of algebraic geometry needed to appreciate that the goofy construction we gave of the representations of $U(n)$ is actually based on some very classical geometry.

1. Two line bundles on projective space

Let $\mathbb{CP}^{n-1} := (\mathbb{C}^n - \{0\})/\mathbb{C}^\times$ denote the space of 1-d subspaces of $\mathbb{C}^n$, or **projective space**. This is naturally a complex manifold.

The incidence variety

$$\widetilde{\mathbb{C}}^n := \{(\vec{v}, l) \in \mathbb{C}^n \times \mathbb{CP}^{n-1} : \vec{v} \in l\},$$

consisting of pairs (vector,line) such that the vector lies in the 1-d subspace, has a natural map to $\mathbb{C}^n$ by projection onto the first factor. Note that if $\vec{v} \neq \vec{0}$, then $l = \mathbb{C}\vec{v}$, so this map is generically one-to-one. But if $\vec{v} = \vec{0}$, the fiber is $\mathbb{CP}^{n-1}$. This is called the **blowup of** $\mathbb{C}^n$ **at the origin**.

It also has a projection onto the second factor. The fiber over an element $l \in \mathbb{CP}^{n-1}$ is exactly the line $l \subseteq \mathbb{C}^n$. This is the **tautological line bundle** over $\mathbb{CP}^{n-1}$.

Now consider the ring R of complex analytic functions on $\widetilde{\mathbb{C}}^n$. By Liouville's theorem, any function on $\mathbb{CP}^{n-1}$ (the fiber over 0) is constant, so these are just the functions on $\mathbb{C}^n$, pulled back. In general we're going to be more interested in the subring of polynomial functions (this being "algebraic" geometry, not complex analytic geometry). That's just a polynomial ring in n variables, but thought of as functions on $\widetilde{\mathbb{C}}^n$.

R is also a representation of the $\mathbb{C}^\times$ that acts on the first factor of $\widetilde{\mathbb{C}}^n$ by rescaling. The weight space R_n for the weight $n \in (\mathbb{C}^\times)^* \cong \mathbb{Z}$ is 0 if $n < 0$, and the homogeneous polynomials of degree n for $n \geq 0$.

Focus now on an element $\vec{v} \in R_1$, which is to say, a linear functional on $\mathbb{C}^n$. Restricted to each fiber of the line bundle $\widetilde{\mathbb{C}}^n$ over $\mathbb{CP}^{n-1}$, this gives a linear function on the line. Equivalently, it gives an element of the dual space to the line.

Working in reverse, if we have a section of the dual line bundle, it gives an element of R_1. We sum this up in a coordinate-free way:

Theorem. *Let V be a complex vector space, and $\mathbb{P}V$ its projective space. Let $\mathcal{O}(1)$ denote the dual line bundle to the tautological bundle $\widetilde{V}$ over $\mathbb{P}V$. Then the space of holomorphic sections of this line bundle is naturally identified with V^*.*

This is relevant for representation theory in the following way:

Theorem. *Let* X *be a (nonempty) variety with a* G*-action,* V *a representation of* G*, and* $i : X \to \mathbb{P}V$ *a* G*-equivariant inclusion. Let* $\mathbb{P}W$ *be the smallest projective-linear subspace of* $\mathbb{P}V$ *containing* $i(X)$*. Then* W *is a nonzero subrepresentation of* V*. Also, the restriction map from sections of* $\mathcal{O}(1)$ *on* $\mathbb{P}V$ *to sections over* $i(X)$ *factors through* $V^* \to W^*$*.*

Proof. $\mathbb{P}W$ is automatically a G-invariant subvariety of $\mathbb{P}V$, since it is defined solely in terms of the G-invariant $i(X)$. When we restrict sections of $\mathcal{O}(1)$ from $\mathbb{P}V$ to $\mathbb{P}W$, that's the map $V^* \to W^*$. Since X is nonempty, $\mathbb{P}W$ is nonempty, so W is not zero. □

2. Borel-Weil: existence

We already noticed that every representation of $U(n)$ is the restriction of an algebraic representation of $GL_n(\mathbb{C})$ (i.e. using only ratios of polynomials).

Proposition. *Let* V *be an irrep of* $GL_n(\mathbb{C})$*. Then there exists a subgroup* P *such that* $GL_n(\mathbb{C})/P$ *is compact, and an algebraic line bundle* L *over* $GL_n(\mathbb{C})/P$*, such that*

- $GL_n(\mathbb{C})$ *acts on the total space of* L
- *the projection* $L \to GL_n(\mathbb{C})/P$ *is* $GL_n(\mathbb{C})$*-equivariant*
- V *is a subspace of the space of algebraic sections of* L*.*

(In fact V will turn out to be *all* the algebraic sections of L, but we can't prove that yet.)

Proof. Look at the action of $GL_n(\mathbb{C})$ on $\mathbb{P}V^*$. Let X be an orbit of minimal dimension. X is necessarily a closed subset of $\mathbb{P}V^*$, because anything in its closure would be of smaller dimension. (One can imagine grosser things happening, like the topologist's sine curve, but these don't happen in algebraic geometry because it only deals with nice functions like ratios of polynomials.)

This orbit X is therefore compact (being a closed subset of projective space, which is compact), and being an orbit, is of the form $GL_n(\mathbb{C})/P$ for P the stabilizer of some point. Let L be the restriction of $\mathcal{O}(1)$ to a line bundle on X.

The restriction of sections of $\mathcal{O}(1)$ over $\mathbb{P}V^*$ to sections over X gives a map from V to the space of sections of L. Since V is irreducible, this map is either an inclusion or zero. But the sections of $\mathcal{O}(1)$ separate the points of projective space, so the map is nonzero. □

It will turn out that we only need to use one P – the space of upper triangular matrices in $GL_n(\mathbb{C})$, usually denoted B for "Borel subgroup". (This is Armand Borel, not his father Émile of Borel measures etc.)

3. Equivariant embeddings of $GL_n(\mathbb{C})/B$ into projective space

So we know we can get the irreps of $GL_n(\mathbb{C})$ by looking at all the ways that $GL_n(\mathbb{C})/P$ sits in projective space (for subgroups P such that $GL_n(\mathbb{C})/P$ is compact). To figure out how to make these for P the upper triangular subgroup B, we need to understand the space $GL_n(\mathbb{C})/B$ better.

3.1. **The flag manifold.** Let $\mathtt{Flags}(\mathbb{C}^n)$ denote the space of maximal chains $F = (\{\vec{0}\} = F_0 < F_1 < \ldots < F_n = \mathbb{C}^n)$ of subspaces of $\mathbb{C}^n$. These chains are called **(complete) flags** and $\mathtt{Flags}(\mathbb{C}^n)$ the **flag manifold** or **flag variety**. (We will see how to make this set into an algebraic manifold in a moment.)

Given a linear transformation $g \in GL_n(\mathbb{C})$, let $g \cdot F = (\{\vec{0}\} < g \cdot F_1 < g \cdot F_2 < \ldots < g \cdot F_n = \mathbb{C}^n)$. This defines an action of $GL_n(\mathbb{C})$ on $\mathtt{Flags}(\mathbb{C}^n)$.

If we take $F_i = \{(*, *, \ldots, *, 0, 0, \ldots, 0)\}$ to be the obvious i-dimensional coordinate subspace, we get the **standard flag**.

Given a flag F, pick unit vectors $\vec{v}_i \in F_i \cap F_{i-1}^{\perp}$ for each $i = 1 \ldots n$. This gives an orthonormal basis from which one can reconstruct each $F_i = \langle \vec{v}_1, \ldots, \vec{v}_i \rangle$. Since $U(n)$ acts (simply) transitively on the set of orthonormal bases, it acts transitively on $\mathtt{Flags}(\mathbb{C}^n)$. This shows that $\mathtt{Flags}(\mathbb{C}^n)$ is naturally a manifold, and compact.

On the other hand, since we have an action of the complex group $GL_n(\mathbb{C})$ on it, and the stabilizer of the standard flag is the complex[1] subgroup B, the quotient is naturally a complex manifold.

3.2. **Grassmannians.** Let $\mathrm{Gr}_k(\mathbb{C}^n)$ denote the space of k-dimensional subspaces of $\mathbb{C}^n$, the k**-Grassmannian**. For much the same reasons as the above, each of these is a compact complex manifold. The most familiar one is $k = 1$, which is just projective space. There is an obvious forgetful map $\mathtt{Flags}(\mathbb{C}^n) \to \mathrm{Gr}_k(\mathbb{C}^n)$ which forgets all the subspaces in a flag other than the kth one. (Q: what are the fibers of this map?)

Multiplying these together, we get an obvious inclusion:

$$\mathtt{Flags}(\mathbb{C}^n) \hookrightarrow \prod_{k=1}^{n-1} \mathrm{Gr}_k(\mathbb{C}^n).$$

So if we can embed Grassmannians into projective space, we can embed flag manifolds into products of projective spaces.

3.3. **The Plücker embedding.** Let $V \hookrightarrow \mathbb{C}^n$ be the inclusion of a k-plane V. Applying the Alt^k functor, we get a nonzero map $\mathrm{Alt}^k V \hookrightarrow \mathrm{Alt}^k \mathbb{C}^n$, with image some one-dimensional subspace of $\mathbb{C}^n$.

Put another way, pick a basis $v_1, \ldots, v_k$ of V; the wedge product $v_1 \wedge v_2 \wedge \ldots \wedge v_k$ defines a nonzero element of $\mathrm{Alt}^k \mathbb{C}^n$.

Either way, this gives a way of taking k-planes in $\mathbb{C}^n$ to lines in $\mathrm{Alt}^k \mathbb{C}^n$, called the **Plücker embedding** $\mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\mathrm{Alt}^k \mathbb{C}^n)$ of the k-Grassmannian. Note that it is naturally equivariant with respect to $GL_n(\mathbb{C})$.

This begins to look like the rep theory – we have the Alt^ks showing up as the basic building blocks.

3.4. **The Veronese embeddings.** The next step is take powers of those blocks. Consider the (nonlinear) map

$$V \to V^{\otimes a}, \qquad \vec{v} \mapsto \vec{v} \otimes \ldots \otimes \vec{v}$$

[1] it's "complex" because it's defined by the vanishing of holomorphic functions, namely the matrix entries in the lower triangle

embedding V in its ath tensor power. It is a fairly unimportant comment that in fact the image lands inside $\mathrm{Sym}^a V$, the symmetric tensors. This map $V \to \mathrm{Sym}^a V$, or rather its projectivization

$$\mathbb{P}V \to \mathbb{P}(\mathrm{Sym}^a V)$$

is called the **ath Veronese embedding**. It is obviously $GL(V)$-equivariant.

So far this gives us a big family of embeddings of flag manifolds into products of projective spaces

$$\mathrm{Flags}(\mathbb{C}^n) \hookrightarrow \prod_{k=1}^{n} \mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \prod_{k=1}^{n} \mathbb{P}(\mathrm{Alt}^k \mathbb{C}^n) \hookrightarrow \prod_{k=1}^{n} \mathbb{P}(\mathrm{Sym}^{a_k}(\mathrm{Alt}^k \mathbb{C}^n))$$

indexed by the choices $\{a_k \in \mathbb{N}\}$.

Foreshadowing. If $k = n$, $\mathrm{Alt}^k \mathbb{C}^n$ and $\mathrm{Sym}^{a_k}(\mathrm{Alt}^k \mathbb{C}^n)$ are is 1-dimensional, the projective space is a point, and it may seem silly to include in the above product. But we'll need it soon.

3.5. **The Segre embedding.** Let V, W be two vector spaces, and consider the (nonlinear) map

$$V \times W \to V \otimes W, \qquad (\vec{v}, \vec{w}) \mapsto \vec{v} \otimes \vec{w}.$$

This descends to the projective spaces, giving the **Segre embedding**

$$\mathbb{P}V \times \mathbb{P}W \hookrightarrow \mathbb{P}(V \otimes W).$$

Now we have our projective embeddings!

$$\mathrm{Flags}(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\otimes_{k=1}^{n} \mathrm{Sym}^{a_k}(\mathrm{Alt}^k \mathbb{C}^n))$$

The discussion from before suggests that we should look for irreps of $GL_n(\mathbb{C})$ as quotients of (the dual space of) $\otimes_{k=1}^{n} \mathrm{Sym}^{a_k}(\mathrm{Alt}^k \mathbb{C}^n)$. But we've run into this already.

4. The relation with rep theory

We've seen this combination $\otimes_{k=1}^{n} \mathrm{Sym}^{a_k}(\mathrm{Alt}^k \mathbb{C}^n))$ before; it was strongly dominated, and so contained an irrep strongly dominated by the same weight. What was unclear in that construction was how to pick out the big irrep from the other junk that showed up in the tensor product.

The geometry makes this separation clearer. The image of the flag manifold in the projective space $\mathbb{P}(\otimes_{k=1}^{n} \mathrm{Sym}^{a_k}(\mathrm{Alt}^k \mathbb{C}^n))$ does not span the whole space; it is contained in a subspace.

Example. Consider the $a_1 = a_2 = 1$ embedding of $\mathrm{Flags}(\mathbb{C}^3)$ into $\mathrm{Gr}_1(\mathbb{C}^3) \times \mathrm{Gr}_2(\mathbb{C}^3) = \mathbb{CP}^2 \times \mathbb{CP}^2$. The coordinates on the first Grassmannian are a, b, c, and on the second are $b \wedge c, c \wedge a, a \wedge b$. Since we're thinking of them as independent we'll call them x, y, z.

The flag manifold is a hypersurface in $\mathrm{Gr}_1(\mathbb{C}^3) \times \mathrm{Gr}_2(\mathbb{C}^3)$, defined by the condition that the line sit in the plane. Another way to say that is that if one picks a basis for the plane, and puts the three lines in a matrix, the matrix will have determinant zero. That equation is $ax + by + cz = 0$.

Now consider the Segre embedding of $\mathbb{CP}^{2=3-1} \times \mathbb{CP}^{2=3-1} \hookrightarrow \mathbb{CP}^{8=3\times 3-1}$. (Sometimes the -1 in the exponent is convenient, sometimes not.) The projective coordinates on the $\mathbb{CP}^8$

are the independent variables $ax, ay, az, bx, by, bz, cx, cy, cz$. (The image is defined by equations like $ax \cdot by = ay \cdot bx$, etc. expressing the fact that these aren't independent on the $\mathbb{CP}^2 \times \mathbb{CP}^2$.) The image of the flag manifold satisfies what is now a *linear* condition: $ax + by + cz = 0$. So the corresponding representation of GL_3 is only 8-dimensional, not 9-dimensional.

(In this case it is even easier to see on the rep side. The rep $\mathrm{Alt}^2\mathbb{C}^3$ (high weight $(1,1,0)$) is isomorphic to the dual $(\mathbb{C}^3)^*$ of the standard representation (high weight $(0,0,-1)$) tensor the determinant representation (high and only weight $(1,1,1)$). So the rep

$$\mathbb{C}^3 \otimes \mathrm{Alt}^2\mathbb{C}^3 \cong \mathbb{C}^3 \otimes (\mathbb{C}^3)^* \otimes \det \cong \mathrm{End}(\mathbb{C}^3) \otimes \det .$$

And this rep $\mathrm{End}(\mathbb{C}^3)$, the conjugation action of $GL_3(\mathbb{C})$ on 3×3-matrices, splits into the scalar matrices and (its perp) the trace-zero matrices. That's the $ax+by+cz = 0$ condition again.)

Anyway, this suggests a way of picking out the irrep we want from $\otimes_{k=1}^{n} \mathrm{Sym}^{a_k}(\mathrm{Alt}^k\mathbb{C}^n))$; understand the algebraic conditions that cut the flag manifold out of the product of Grassmannians (this plane's in that plane), that cut out the Veronese, that cut out the Plücker embedding (the "Plücker relations"), and that cut out the Segre (the $ax \cdot by - ay \cdot bx$-type from above). We won't do this, but it can be done, and is done in detail in Fulton's Young Tableaux.

Technical note. It seems like the $\{a_k\}$ all had to be naturals, so we're getting a pointy cone's worth $(\mathbb{N}^n)$ of representations here. But we know that the cone of dominant weights is $\cong (\mathbb{N}^{n-1}) \times \mathbb{Z}$. What gives?

We know the answer: we need negative powers of the determinant. Which is to say, that last power a_n can be negative. It's only involved in a symmetric power of a 1-dimensional space.

Geometrically, this can look like a pretty weird thing to worry about – isn't this eventually just affecting our choice of a *point*, $\mathbb{P}(\mathrm{Sym}^{a_n} \det)$? But no, because over that point is a line bundle (really, just a line), and the choice of a_n affects how the group $GL_n(\mathbb{C})$ acts on that line. So eventually, when we pull the line bundle back and take sections of it, the representation of $GL_n(\mathbb{C})$ will be different depending on the choice of a_n.