GROUP THEORY IN PHYSICS: AN EXAMPLE
SUMMARY FOR MATH 261, FALL 2001

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1. The warmup problem

1.1. The input from physics. Consider a mass connected to one end of a spring, free to move along a line, the other end fixed to a wall. Denote by $x(t)$ the difference between the position of the mass at time t and its rest position (so $x(t) \equiv 0$ describes a stationary mass). Then $x(t)$ satisfies the differential equation

$$\frac{d^2}{dt^2}x = -kx$$

at least so long as $x(t)$ stays small enough that Hooke's law[1] applies. The constant k here is determined by the mass and the spring constant.

Next, consider two particles in the plane, connected by a spring, with displacements $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ (again functions of time t) from a stationary position in which they would ordinarily be separated by $\vec{v}$. These now satisfy the coupled differential equations

$$\frac{d^2}{dt^2}\vec{x} = -\frac{\vec{v} - \vec{x} + \vec{y}}{|\vec{v} - \vec{x} + \vec{y}|}(|\vec{v} - \vec{x} + \vec{y}| - |\vec{v}|)$$
$$\frac{d^2}{dt^2}\vec{y} = -\frac{\vec{v} - \vec{y} + \vec{x}}{|\vec{v} - \vec{y} + \vec{x}|}(|\vec{v} - \vec{y} + \vec{x}| - |\vec{v}|)$$

but again, let's take the trick of linearizing nearby $\vec{x}, \vec{y} = \vec{0}$. Then these become

$$\frac{d^2}{dt^2}\vec{x} = -\frac{\vec{v}}{|\vec{v}|}(\vec{v} \cdot (\vec{x} - \vec{y}))$$
$$\frac{d^2}{dt^2}\vec{y} = -\frac{\vec{v}}{|\vec{v}|}(\vec{v} \cdot (\vec{y} - \vec{x}))$$

which is linear. This is only an approximation, of course, appropriate for small perturbations around a stable position (and even then it can be problematic as we will see soon).

1.2. The matrix formulation. Let $\vec{w} = [x_1\ x_2\ y_1\ y_2]^T$, and assume now that the rest positions of our particles are at $(0, 0)$ and $(1, 0)$ (i.e. the rest displacement is $\vec{v} = (1, 0)$). Then these coupled differential equations say

$$\frac{d^2}{dt^2}\vec{w} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{w}$$

[1] This is the premiere example of physicists taking the calculus observation that differentiable functions can be locally approximated by linear functions to first order and calling it a "law".

(this is easy to calculate: the i, jth entry says how a displacement in the ith coordinate of $\vec{w}$ pulls or pushes the jth coordinate). Call this matrix H.

1.3. **Rep theory enters.** This problem has a couple of symmetries: we can reflect up/down

$$r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and we can reflect left/right *while switching the identities of the particles*,

$$s = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

These make our 4-d space into a representation of $Z_2 \times Z_2 = \langle r, s \rangle$. The statement that these are symmetries of the problem is simply that H commutes with this group.

My philosophy of this: r and s give ways for two different observers to convert between one another's view of the same system. Since physics is the same for the both of them, this conversion process should commute with time evolution (here determined by H).

1.4. **The character.** The character of this representation is $\mathrm{Tr}\, e = 4$, $\mathrm{Tr}\, r = \mathrm{Tr}\, s = \mathrm{Tr}\, rs = 0$. In fact it is the regular representation of $Z_2 \times Z_2$, so decomposes into one copy each of the irreps.

(We are very lucky in this example, in that we have so much symmetry. If we had only noticed e.g. the relabeling of the particles, we would have two copies of each irreducible for Z_2.)

We can decompose into isotypic components using e.g. the projection formula

$$\pi_{W,V} = \frac{\dim V}{|G|} \sum_g \overline{\mathrm{Tr}\,(g|_V)}\, g|_W.$$

Note that each of these is automatically invariant under H, by Schur's lemma, and since they're actually irreducible in this case H acts as a scalar on each.

In this case they are spanned by the vectors

$$[1\ 0\ -1\ 0]^T \qquad \text{the trivial representation}$$

Pull the particles apart, and they bounce in and out. The H eigenvalue is -2, the sign indicating the restoring force involved.

$$[0\ 1\ 0\ 1]^T \qquad r = 1, s = -1$$

Pull them both up. The H eigenvalue is 0; this is stable.

$$[1\ 0\ 1\ 0]^T \qquad r = -1, s = 1$$

Pull them both left. The H eigenvalue is 0; this is stable.

$$[0\ 1\ 0\ -1]^T \qquad r = -1, s = -1$$

Pull one up, one down (start the spring rotating). The H eigenvalue is 0. In this case, of course, the particles will not continue forever in straight lines; this is where the linear approximation breaks down.

2. More springs

Now consider three particles on the line, connected by two springs, with displacements x_1, x_2, x_3 from rest positions. The corresponding H-matrix is the sum of two:

$$\frac{d^2}{dt^2}\vec{w} = \left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \right) \vec{w}$$

We again have a left-right reflection symmetry, switching the identities of the first and third particles:

$$r = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

When we decompose into irreps of $Z_2 = \langle r \rangle$ (which is to say, eigenspaces for r), something new happens. From character analysis (Tr $e = 3$, Tr $r = -1$) we know we'll get one copy of the trivial representation and two of the sign representation:

$$\frac{1}{2}(3, -1) \cdot (1, 1) = 1, \quad \frac{1}{2}(3, -1) \cdot (1, -1) = 2.$$

H will have to preserve each of those isotypic components. And on the first one, since it's irreducible, it will have to act as a scalar. But on the second it may not (and in this case it won't).

The invariant vectors (the trivial-rep isotypic component) are $\{[a \; 0 \; -a]^T\}$, which correspond to pulling the two outer particles apart, and leaving the middle one alone. The H-eigenvalue is -1, since on each outer particle only one spring is involved.

The sign-rep isotypic component is spanned by $[1 \; 1 \; 1]^T$ and $[1 \; -2 \; 1]^T$. These are eigenvectors of H, with eigenvalues 0 (no restoring force when all are pulled sideways) and -3 (the strongest force when the middle one is pulled the opposite direction from the other two). In particular H does *not* act on this isotypic component as a scalar.

3. The tetrahedron

Now consider four particles in 3-space, at the vertices of a tetrahedron, connected by six springs. The handy places to put them at rest are

$$(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1).$$

Consider the spring between the first two particles. The displacement is $\vec{v} = (1, 1, 0)$. So the corresponding H is

$$H_{1,2} = \begin{bmatrix} -1 & -1 & 0 & 1 & 1 & 0 \\ -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Each of those columns matches the displacement $(1, 1, 0)$ since the spring can only push/pull along itself (in our small-displacement assumption, that the springs don't really rotate), and the size $1, -1$, or 0 comes from whether a small displacement stretches or contracts the spring.

We could write down the whole H, but rather than do that, let's think about the action of S_4 on our space $\mathbb{C}^{12}$. It acts by permuting the identities of the particles, and at the same time, on the 3-space containing the particles.

What is the action on that 3-space? If we imagine the origin lying at the center of the tetrahedron, and moving all the particles slightly off into a fourth dimension, we see that we're thinking about the standard action of S_4 on $\mathbb{C}^4$. Without that slight push into the fourth dimension, this is S_4 on the sum-zero part of $\mathbb{C}^4$.

What then is the action on the 12-space? This will be by 12×12 matrices that break up into sixteen 3×3 blocks, such that in the large they look like permutation matrices, but up close (in one of those blocks) they look like the 3-space action just discussed. In other words, they are the Kronecker products of the permutation matrices and the 3×3 matrices just discussed. As such the trace is the product of the two traces.

The character table of S_4 is

1^4	$2, 1^2$	$3, 1$	4	$2, 2$
1	1	1	1	1
1	-1	1	-1	1
2	0	-1	0	2
3	1	0	-1	-1
3	-1	0	1	-1

(where the head of each column is labeled by the cycle structure of the permutation). The character of the representation in question is $(12, 2, 0, 0, 0)$. By computing dot products of that with the rows of the character table, we determine that this breaks up as one copy of the trivial irrep, none of the sign irrep, one of the 2-d irrep $\mathbf{2}$, two of the first 3-d irrep $\mathbf{3}$, and one of the second 3-d irrep $\mathbf{3}'$. Note that $\mathbf{3}' \cong \mathrm{Alt}^2 \mathbf{3}$.

(All but one of our isotypic components are in fact irreducible. So once we decompose into isotypic components, we've practically diagonalized H.)

At this point we're trying to visualize actual vectors in each of these isotypic components. Since these isotypic components are not 1-dimensional, this is akin to picking a basis for them, and there is no God-given way to do this. Our approach will be to look only for a spanning set, and ask that it be extremely symmetric (each vector invariant under a large subgroup of S_4).

Consider the decomposition of each of the irreps under S_3. To see the S_3-character, forget the last two columns.

- The trivial rep stays trivial.
- The sign rep is again a sign rep.
- The $\mathbf{2}$ remains irreducible.
- The $\mathbf{3}$ is the sum of the 2-d rep and a trivial rep.
- The $\mathbf{3}'$ is the sum of the 2-d rep and a sign rep.

In particular, there is only a 3-d space of S_3-invariant vectors, and they are easy to visualize, if you look in toward the center from the position of vertex #4 (the S_3-fixed one). That one must be moving either directly forward or backward (in or out toward the center), and #i moving in the i-4-center plane, for $i = 1, 2, 3$, all three of them moving the same way. That's one degree of freedom for the motion of particle #4, and two degrees for the motion of particle #1, but then that determines the motion of particles #2 and #3.

The S_4-invariant mode is obvious. All four particles are moving directly in or out the same amount. This has the most negative H-eigenvalue.

One **3**-rep is also fairly obvious, since **3** is the representation of S_4 on the ambient space. "The four particles all moving in the same direction" describes a 3-d invariant subspace isomorphic to **3**. This stretches no springs and has H-eigenvalue zero.

To get an understanding of the other **3**-subrep, look at the S_3-invariant vector that is perp to the other two S_3-invariant vectors just described. (Since H is symmetric, its eigenvectors for different eigenvalues are all orthogonal.) Particle #4 comes towards us, the other three move away, but contract toward one another. Hitting this vector with S_4, we get four vectors that sum to zero.

Rotations are described by angles of rotation through planes. The number of coordinate planes in n-dimensional space is $\binom{n}{2}$, the dimension of $\mathrm{Alt}^2\mathbb{C}^n$. This is a halfhearted attempt to convince you that the subspace of infinitesimal rotations (obviously S_4-invariant) is the $\mathbf{3}'$ subrep.

To describe the remaining modes (in the **2**) is only a little more subtle, and we will use D_4 for it. Its character table is

e	$\{(13)(24)\}$	$\{(12)(34),(14)(23)\}$	$\{(13),(24)\}$	$\{(1234),(1432)\}$
1	1	1	1	1
1	1	1	−1	−1
1	1	−1	1	−1
1	1	−1	−1	1
2	−2	0	0	0

If we decompose the irreps of S_4 under D_4, we find

- The trivial rep remains trivial.
- The sign rep turns into row two of this table.
- The **2** becomes the trivial rep plus row two.
- The **3** becomes the bottom row plus the third row.
- The $\mathbf{3}'$ becomes the bottom row plus the fourth row.

In particular, only the trivial rep and the **2** of S_4 have D_4-invariant vectors. It is easy to visualize one of the D_4-invariant vectors: pull particles #1 and #3 directly toward each other, likewise #2 and #4, everyone moving the same amount. This can be written as some amount of the breathing mode plus a vector perp to the breathing mode, which by the above must lie in the **2**.

Again, when we hit this displacement by S_4, we get three different vectors whose span is the **2**, and there is no particularly good way to choose a basis.