WEYL GROUPS AND WEYL CHAMBERS
NOTES FOR MATH 261, SPRING 2002

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Let K be a compact connected group, T a maximal torus, $W := N(T)/T$ the Weyl group, Δ the root system, and Λ the weight lattice of T. Then W acts on T, $\mathfrak{t}$, and Λ. If we pick a K-invariant inner product on $\mathfrak{t}$, it induces a W-invariant one on all these other spaces.

Recall that a **positive system** $\Delta_+ \subseteq \Delta$ is the intersection of Δ with a half-space in $\mathfrak{t}^*$, such that the boundary hyperplane contains no roots. If we twist the half-space by a Weyl group element, we get another positive system.

We already know some good elements of W; for any root $\beta \in \Delta$, we have the reflection r_β negating β, and preserving the hyperplane $\beta^\perp$. If β is a simple root, we call r_β a **simple reflection**.

Lemma. *Let $\alpha \in \Delta_1$ be a simple root. The difference between Δ_+ and $r_\alpha \Delta_+$ is just that the first contains α and the second contains $-\alpha$.*

Proof. Recall first that every root is either a positive sum of simple roots, or a negative sum, and not both (since they're linearly independent).

If $\beta \in \Delta_+$ is not α, then it is not a multiple of α, and therefore contains some other simple root with a positive coefficient. When we reflect using r_α, it doesn't change that positive coefficient. Therefore the root stays positive. So the only root that goes negative is α, to $-\alpha$. $\square$

Proposition. *The Weyl group acts transitively on the space of positive systems. Actually, this is already true of the subgroup generated by simple reflections. (This subgroup will be proven later to be the whole group.)*

Proof. Let P be a positive system, which we want to bring to Δ_+. If P contains all of the simple roots Δ_1, then $P \supset \Delta_+$, so $P = \Delta_+$. Contrapositively, if $P \neq \Delta_+$, there exists some simple root α such that $-\alpha \in P$.

We claim that $\Delta_+ \cap (r_\alpha \cdot P)$ is strictly larger than $\Delta_+ \cap P$, since

$$\Delta_+ \cap (r_\alpha \cdot P) = r_\alpha \cdot ((r_\alpha \cdot \Delta_+) \cap P) = r_\alpha \cdot ((\Delta_+ \setminus \{\alpha\} \cup \{-\alpha\}) \cap P)$$

$$= r_\alpha \cdot ((\Delta_+ \cap P) \cup \{-\alpha\}) = (r_\alpha \cdot (\Delta_+ \cap P)) \cup \{\alpha\}$$

This last union is disjoint, since $\alpha \not\ni r_\alpha \Delta_+$.

So we can move P using simple reflections to increase its intersection with Δ_+, so by induction we can move it all the way to Δ_+. Therefore the subgroup generated by reflections acts transitively on the set of positive systems. $\square$

In particular there exists an element $w_0 \in W$, called the **long word** in the Weyl group, with the property $w_0 \cdot \Delta_+ = \Delta_-$. In $GL_n(\mathbb{C})$'s Weyl group S_n this was $n, n-1, \ldots, 3, 2, 1$.

Warning: we will casually use w_0 to denote an element of G in what follows. This isn't fair, because it really lives in $N(T)/T$, with no canonical lift in $N(T)$. So it's up to you to check that each place we use it (such as to speak of "$w_0 B \in G/B$") the choice of lift is immaterial.

Lemma. *The N-orbit through the point $w_0 B \in G/B$ is free ($n \mapsto n w_0 B$ is an injection), and open dense.*

Its image in G/B is called the **big cell**, "cell" in the topological sense since N is a vector space.

Proof. The group $N_- := \exp(\sum_{\Delta_-} \mathfrak{g}_\beta) = w_0 N w_0^{-1}$, generated by the negative root spaces, does not intersect B. Proof: we already know that N_- injects under the adjoint representation, and N_- maps to lower unipotent matrices whereas B maps to upper triangulars.

This shows the injectivity of $n \mapsto n w_0 B$, because if $n w_0 B = w_0 B$, then $w_0^{-1} n w_0 \in B$, so it's in $N_- \cap B = \{1\}$.

For the openness, then, we only need know that $\dim N = \dim G/B$, which is obvious on the Lie algebra level (since $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}_-$).

For the density, note that N is complex and is acting holomorphically on G/B, so each orbit-type stratum is complex, so even real dimensional. Hence ripping out the lower-dimensional strata does not disconnect G/B. (Of course, we need G/B connected to begin with, i.e. G connected.) ☐

Theorem. *The Weyl group acts* simply *transitively on the set of positive systems. Also, the Weyl group is generated by reflections.*

Proof. We first show that the N-orbit $N w_0 B$ contains no other wB, $w \in W$; in particular $N w_0 B \neq N w B$ for any other $w \in W$. This is because the map $\mathfrak{n} \to G/B$, $X \mapsto \exp(X) w_0 B$ is T-equivariant and injective, so the only T-fixed point in the image comes from the unique T-fixed point in $\mathfrak{n}$, namely $\vec{0}$. And we already knew the T-fixed points on G/B, because we'd figured them out on K/T.

To show the freeness of W's action on the set of positive systems, we just need to show that there exists a positive system Γ with a property such that $w \cdot \Gamma$ does not have that property for any $w \neq 1$. The property we use is "having a dense N-orbit", and the Γ we use is $\Delta_- = w_0 \cdot \Delta_+$. *Since* G/B *is connected* (see comment following proof), it can't have two open dense orbits. And we just showed above that the orbit NwB isn't $N w_0 B$ unless $w = w_0$.

Therefore there's at most one Weyl group element taking any positive system to any other. But we already showed that there *is* a Weyl group element, generated by reflections, taking one to any other. Therefore there is exactly one, and it's generated by reflections. ☐

It seems a shame to go to such a noncombinatorial proof, but it's necessary; the statement can be false if G isn't connected.

Exercise. Let Z_2 act on $SU(3)$ by $M \mapsto M^*$. Show that the theorem fails for the semidirect product $Z_2 \ltimes SU(3)$.

Exercise. Use the theorem to show that for any subgroup $P > B$, the dimensions can't agree.

Given a positive system $\Gamma \subseteq \Delta$, define the **open Weyl chamber** as the set of $X \in \mathfrak{t}$ such that $X \cdot \Gamma \in \mathbb{R}_+$. This is actually an open polyhedral cone, isomorphic to $\mathbb{R}_+^n$, whose walls come from the hyperplanes perpendicular to the simple roots. (Having positive dot product with the simple roots is enough to guarantee positive dot product with all the positive roots.)

Plainly two positive systems have disjoint Weyl chambers, and almost every element of $\mathfrak{t}$ is in a Weyl chamber. The only way to go wrong is to be in one of the hyperplanes perpendicular to a root.

Put another way, one can start with $\mathfrak{t}$, rip out all the hyperplanes $\alpha^\perp$, and the components of the complement are the Weyl chambers.

Theorem. *W acts simply transitively on the set of open Weyl chambers. If we pick a particular one, the "**positive Weyl chamber**" $\mathfrak{t}_+$, and take its closure, then every element of $\mathfrak{t}$ is W-conjugate to a unique element of $\mathfrak{t}_+$.*

For this reason "Weyl chamber" usually means the closed chamber.

Note that we can perform this same decomposition on $\mathfrak{t}^*$, without using an inner product – for each reflection r_α, rip out the hyperplane perpendicular to α, pick one of the resulting components, and call its closure the positive Weyl chamber in $\mathfrak{t}^*$. This is where we mostly want them, of course, since this is where weights of T live. We now give a correspondence between positive systems (which are subsets of $\mathfrak{t}^*$) and Weyl chambers in $\mathfrak{t}^*$, without choosing an identification of $\mathfrak{t}$ and $\mathfrak{t}^*$.[1]

Define the **Weyl vector** ρ as $\frac{1}{2}\sum_{\Delta_+} \alpha$, one-half the sum of the positive roots. The $\frac{1}{2}$ is desirable, because of the following equation (a consequence of the first lemma):

$$r_\alpha \cdot \rho = \rho - \alpha.$$

Exercise. What is this ρ in the $GL_n(\mathbb{C})$ case? And in what context did we see it (perhaps with a constant added)?

Lemma. *The Weyl vector ρ has no W-stabilizer.*

Proof. Let $X \in \mathfrak{t}$ have positive pairing with each simple root (and therefore all positive roots). Let $w \in W$ be some nonidentity element of the Weyl group. Let $D = \Delta_+ \cap (w \cdot \Delta_+)$. Then $\Delta_+ \setminus D$ consists only of positive roots (and is nonempty), and $(w \cdot \Delta_+) \setminus D$ consists only of negative roots (and is nonempty).

Now we claim that $\langle X, \rho \rangle$ is strictly more than $\langle X, w \cdot \rho \rangle$, since the difference is one-half

$$\sum_{\Delta_+ \setminus D} \langle X, \alpha \rangle - \sum_{(w \cdot \Delta_+) \setminus D} \langle X, \alpha \rangle > 0$$

and in particular, $\rho \neq w \cdot \rho$. □

Proposition. *There is a canonical correspondence between positive systems of roots and Weyl chambers in $\mathfrak{t}^*$.*

[1]This is usually blithely skipped over by using the Killing form to define an inner product on $\mathfrak{k}$, and then restrict to $\mathfrak{t}$. Unfortunately, the Killing form can be degenerate (e.g. if $K = T$), so it's better to have a construction like the one here, that doesn't use it.

Proof. Since ρ has no stabilizer, it does not lie on any of the root hyperplanes, and therefore lies in a unique Weyl chamber. This gives a way of associating a Weyl chamber in $\mathfrak{t}^*$ to a positive system. □

Culture: there are some infinite-dimensional Lie groups with finitely many simple roots but infinitely many positive roots. In this case the infinite sum in our definition of the Weyl vector doesn't make sense. It is still useful to define one, but it has to be characterized by $r_\alpha \cdot \rho = \rho - \alpha$.