MATH 2230 MIDTERM, FALL 2014

Name, **written slowly and legibly:** ______________________________

> **Added after the test:** I forgot to *give points* for the above. If I had, they would definitely have been *taken away* from some people. I would say "you know who you are" but I can only assume they don't!
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> In the below I've put in [editorial comments] that I wouldn't expect people to actually write in an answer; they're there to help you understand how to come up with an answer.

In each answer, write as much (on front and back) as it takes to convey your thought process; full English sentences are much easier to give credit to than bare, unmotivated scribbled formulæ. (They won't do any good if they can't be read, so *do* put effort into making them legible.)

Feel free to ask me questions during the test, especially if you need a little reminder about a definition. Worst case is I don't answer. (It's very sad to afterward hear "I didn't realize I could ask you that" — find out!)

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1. Let g be the ABACAB function of three $n \times n$ matrices:

$$g(A, B, C) = ABACAB$$

a [5 pts]. If you wrote down the Jacobian of g – but don't! – how big a matrix would it be (what by what)?

Answer. $3n^2 \times n^2$.

[g eats three $n \times n$ matrices and produces one.]

1b [15 pts]. Compute the derivative of g, at every (A, B, C). Your answer should (had better) be a linear transformation of its inputs.

Answer. Let's perturb by (H_A, H_B, H_C). Then

$$\lim_{\varepsilon \to 0} \frac{g(A + \varepsilon H_A, B + \varepsilon H_B, C + \varepsilon H_C) - g(A, B, C)}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{ABACAB + \varepsilon(H_A BACAB + AH_B ACAB + ABH_A CAB + ABAH_C AB + ABACH_A B + ABACAH_B)}{\varepsilon}$$

$$= H_A BACAB + AH_B ACAB + ABH_A CAB + ABAH_C AB + ABACH_A B + ABACAH_B$$

i.e. the derivative at (A, B, C) is the linear transformation

$$(H_A, H_B, H_C) \mapsto H_A BACAB + AH_B ACAB + ABH_A CAB + ABAH_C AB + ABACH_A B + ABACAH_B$$

[Sanity check: consider $n = 1$ where things commute, so $g(A, B, C) = A^3B^2C$, and this would give $3A^2B^2C\ H_A + 2A^3BC\ H_B + A^3B^2\ C$ Then the partial derivatives are $\frac{dg}{dA} = 3A^2B^2C$, $\frac{dg}{dB} = 2A^3BC$, $\frac{dg}{dC} = A^3B^2$, which are indeed the coefficients in front of H_A, H_B, H_C.]

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2. Let $C \subseteq \mathbb{R}^n$ be an open set containing $\vec{0}$, and for $\lambda \in \mathbb{R}_+$ and $\vec{u} \in \mathbb{R}^n$, define

$$\vec{u} + \lambda C := \{\vec{u} + \lambda\vec{c} \; : \; \vec{c} \in C\}.$$

Call a set $U \subseteq \mathbb{R}^n$ C-**open** if for every $\vec{u} \in U$, there exists a $\lambda > 0$ such that $U \supseteq \vec{u} + \lambda C$.

a [10 pts]. If U is C-open, prove U is open.

Answer. We need to show that for any $\vec{u} \in U$, there exists a ball $B_{r>0}(\vec{u}) \subseteq U$. And by the way that's $\vec{u} + rB_1(\vec{0})$.

What we're given is that there exists a $\vec{u} + \lambda C \subseteq U$. So if we can find a ball $\vec{u} + rB_1(\vec{0}) \subseteq \vec{u} + \lambda C$, i.e. $rB_1(\vec{0}) \subseteq \lambda C$, then we're done.

Since C is open and $\vec{0} \in C$, we know there's a ball $B_s(\vec{0}) \subseteq C$. Scaling the vectors in both, we know $\lambda B_s(\vec{0}) \subseteq \lambda C$. So if we can find a ball $rB_1(\vec{0}) \subseteq \lambda B_s(\vec{0})$, then we're done.

That's easy: take $r = \lambda s$. Then those balls are equal.

2b [20 pts]. If C is bounded, and U is open, prove U is C-open.

Answer. We need to show that for any $\vec{u} \in U$, there exists a $\lambda > 0$ with $\vec{u} + \lambda C \subseteq U$.

We're given that C is bounded, i.e., there exists a radius $R > 0$ such that $C \subseteq B_R(\vec{0})$. Also that U is open, so there exists a ball $B_s(\vec{u}) \subseteq U$ for some $s > 0$.

So we'd like to choose λ so that $\vec{u} + \lambda C \subseteq B_s(\vec{u})$. We know $\vec{u} + \lambda C \subseteq \vec{u} + \lambda B_R(\vec{0}) = B_{\lambda R}(\vec{u})$, so it's enough to get $B_{\lambda R}(\vec{u}) \subseteq B_s(\vec{u})$. As in (2a), that's easy: take $\lambda = s/R$.

2c [15 pts]. Give an example of a C and an open set U that isn't C-open.

Answer. By (2b), we need C unbounded. The dumbest example is $C = \mathbb{R}^n$ (for $n > 0$). Then *every* $\vec{u} + \lambda C = \mathbb{R}^n$, too. So as long as $U \neq \emptyset, \mathbb{R}^n$, it's a counterexample [but you can't stop there! e.g. for n there is no such U]; let's say $U = (0, 1) \subseteq \mathbb{R}^1$.

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3. Let A be an $n \times n$ matrix.

For $S \leq \mathbb{R}^n$ a linear subspace, define

$$S^\bullet := \{\vec{v} \in \mathbb{R}^n \; : \; \forall \vec{s} \in S, \; \vec{s} \cdot A\vec{v} = 0\}$$

a [10 pts]. Prove that $S^\bullet$ is a linear subspace too. (Meaning: show that it satisfies the short list of requirements.)

Answer. There are three requirements: it should have $\vec{0}$, be closed under multiplication by any scalar, and closed under addition.

$$\forall \vec{s} \in S, \; \vec{s} \cdot A\vec{0} = \vec{s} \cdot \vec{0} = 0 \quad \checkmark$$

If $\vec{v} \in S^\bullet$ and $c \in \mathbb{R}$, then $\forall s \in S$, we have

$$\vec{s} \cdot A(c\vec{v}) = \vec{s} \cdot cA\vec{v} = c \; \vec{s} \cdot A\vec{v} = c0 = 0.$$

So $c\vec{v} \in S^\bullet$ too. ✓

If $\vec{v}_1, \vec{v}_2 \in S^\bullet$, then $\forall s \in S$, we have

$$\vec{s} \cdot A(\vec{v}_1 + \vec{v}_2) = \vec{s} \cdot (A\vec{v}_1 + A\vec{v}_2) = \vec{s} \cdot A\vec{v}_1 + \vec{s} \cdot A\vec{v}_2 = 0 + 0 = 0.$$

So $\vec{v}_1 + \vec{v}_2 \in S^\bullet$ too. ✓

3b [15 pts]. If A is symmetric, prove that $(S^\bullet)^\bullet \geq S$. (Meaning: assume $\vec{b} \in S$, and prove $\vec{b} \in (S^\bullet)^\bullet$.)

Answer. Assume $\vec{b} \in S$. We want to know that $\vec{b} \in (S^\bullet)^\bullet$, i.e. if $\vec{t} \in S^\bullet$, then $\vec{t} \cdot A\vec{b} = 0$. What we know for sure is that $\vec{b} \cdot A\vec{t} = 0$, since $\vec{t} \in S^\bullet$.

[Those two equations say different things about $\vec{b}$, since it's on the right in one and the left in the other. That, and the fact that we're given $A = A^T$, suggest using transpose.]

We can rewrite $\vec{b} \cdot A\vec{v}$ as the entry of the 1×1 matrix $\vec{b}^T A\vec{v}$, thinking of $\vec{b}, \vec{v}$ as skinny matrices. Then since a 1×1 matrix is symmetric,

$$\vec{b}^T A\vec{v} = (\vec{b}^T A\vec{v})^T = \vec{v}^T A^T \vec{b}$$

then use $A = A^T$ and learn $\vec{b} \cdot A\vec{v} = \vec{v} \cdot A\vec{b}$.

Since $\vec{b} \cdot A\vec{t} = 0$, and $\vec{b} \cdot A\vec{v} = \vec{v} \cdot A\vec{b}$, we learn $\vec{t} \cdot A\vec{b} = 0$ for all $t \in S^\bullet$, i.e. $\vec{b} \in (S^\bullet)^\bullet$.

3c [10 pts]. Give an example of n, A symmetric, and S such that $(S^\bullet)^\bullet \neq S$.

Answer. $n = 1$, $A = [0]$, $S = \{\vec{0}\}$. Then $S^\bullet = (S^\bullet)^\bullet = \mathbb{R}^1 > S$.

[Thought process: we know $(S^\bullet)^\bullet > S$, by (3b). For it to grow in this way, we need being-in-$V^\bullet$ to be easy for $V = S, S^\bullet$. The $= 0$ condition in the definition of $V^\bullet$ gets easier to satisfy as $A\vec{v}$ gets to be $\vec{0}$ more often. The easiest way to ensure that is to take A the zero matrix.

In fact, $(S^\bullet)^\bullet = S$ for all $S \leq \mathbb{R}^n$ iff A is invertible.]