MATH 2240 MIDTERM, SPRING 2015

Name, **written slowly and legibly:** ________________________________

In each answer, write as much (on front and back) as it takes to convey your thought process; full English sentences are much easier to give credit to than bare, unmotivated scribbled formulæ. (They won't do any good if they can't be read, so *do* put effort into making them legible.)

Feel free to ask me questions during the test, especially if you need a little reminder about a definition. Worst case is I don't answer. (It's very sad to afterward hear "I didn't realize I could ask you that" — find out!)

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1. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded function, and $I_1 = [a_1, b_1), I_2 = [a_2, b_2), \ldots$ a sequence of intervals.

a [15 pts]. If f is Riemann integrable, show that for any $\epsilon > 0$, there exists such a sequence of intervals such that

$$\sum_k 1_{I_k} \mathrm{inf}_{I_k}(f) \leq f \leq \sum_k 1_{I_k} \mathrm{sup}_{I_k}(f) \qquad \text{pointwise}$$

and

$$\sum_k (b_k - a_k)(\mathrm{sup}_{I_k}(f) - \mathrm{inf}_{I_k}(f)) < \epsilon$$

where $\mathrm{inf}, \mathrm{sup}$ are the greatest lower and least upper bounds of f on I_k.

Answer. If f is Riemann integrable, it's of bounded support inside some interval $[-2^n, 2^n)$, and we can take for (I_k) the finite sequence of N-dyadic intervals inside there.

By the definition of Riemann integrability, as $N \to \infty$ the difference in this second line goes to 0.

1b [10 pts]. Show the converse of (a) fails, i.e., find a bounded function f such that for any ϵ we have such a sequence of intervals, but f isn't Riemann integrable.

Answer. Let $f(x) = 58$ for all x, and $I_k = [n_k, n_k + 1)$ where (n_k) is an enumeration of the integers. Then the inequalities are strict and the second sum is 0. But of course f isn't integrable.

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2. Let f be Riemann integrable. Define g by

$$g(x) = \sum_{n \in \mathbb{N}} \frac{f(2^n x)}{2^n}$$

a [10 pts]. Show g is Lebesgue integrable.

Answer. By the change of variable formula, $\int |f(2^n x)| = \frac{1}{2^n} \int |f(x)|$, so $\sum_n \int |f(2^n x)| = (\sum_n \frac{1}{4^n}) \int |f(x)| < \infty$. Then use our definition of Lebesgue-integrable function.

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b [20 pts]. Show g is *Riemann* integrable. (Hint: don't use the definition; we have a killer theorem about when this is true.)

Answer. If f is supported within a ball of radius r, so is g.

If $|f| \leq c$ everywhere, then $|g(x)| \leq \sum_n \frac{c}{2^n} \leq 2c$.

If $S(f)$ is the set of points where f is discontinuous, then we claim that

$$S(g) \subseteq (S(f) \ \cup \ S(f)/2 \ \cup \ S(f)/4 \ \cup \ \ldots) \ \cup \ \{0\}$$

Why? Outside of any ball of radius ϵ, $g = \sum_{n \leq \log_2(r/\epsilon)} \frac{f(2^n)}{2^n}$, so can only be discontinuous where those functions are. Since each of those sets $S(f)/2^k$ is measure zero and there are only countably many of them, $S(g)$ is measure zero.

Hence g is bounded with bounded support and continuous almost everywhere, so Riemann integrable.

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3 [15 pts]. Find a **continuous** function $f(x, y)$ on $\mathbb{R}^2$ such that the iterated Riemann integrals

$$\int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} f(x, y)$$

exist but the iterated Riemann integrals

$$\int_{y \in \mathbb{R}} \int_{x \in \mathbb{R}} f(x, y)$$

don't, i.e. *something* goes wrong in the latter. Did I mention f should be **continuous?**

Answer.

$$f(x, y) = \begin{cases} \sin(y) & \text{if } y \in [0, 2\pi] \\ 0 & \text{otherwise} \end{cases}$$

Then each $\int_{y \in \mathbb{R}} f(x, y) = 0$, and those 0s can be integrated over x, too. But if we attempt $\int_{x \in reals} f(x, y)$ for $y = \pi/2$ say, we're looking at $\int_{x \in reals} 1$ which isn't integrable.

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4. Define $\Phi : (0, 1) \to (1, \infty)$ by $\Phi(x) = \frac{1}{x}$.

a [10 pts]. Given a function $f : (1, \infty) \to \mathbb{R}$, write down the change-of-variable formula for integrating f vs. $f \circ \Phi$.

Answer. $D\Phi|_x = \left[\frac{-1}{x^2}\right]$, so $|\det D\Phi|_x| = \frac{1}{x^2}$. Hence

$$\int_{x \in (0,1)} \frac{f(\Phi(x))}{x^2} = \int_{y \in (1,\infty)} f(y).$$

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b [10 pts]. Find a function f such that exactly one of the integrals from part (a) exists as a Riemann integral. (The other side will only be Lebesgue integrable.)

Answer. Let's make the left integral be of 1, i.e. $f(\Phi(x))/x^2 = 1$. Then $x^2 = f(\Phi(x)) = f(1/x)$, or $1/y^2 = f(y)$. Now the RHS is $\int_{y \in (1,\infty)} \frac{1}{y^2}$, which isn't of bounded support so isn't a Riemann integrable function. But the LHS is fine.

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c [10 pts]. In part (b), why can't you make the *other* one of the two integrals be the only one that exists as a Riemann integral?

Answer. Say the RHS exists as a Riemann integral. Then f is of bounded support, and in particular $f(y) = 0$ for $y > b$ (for some b). Hence $(f \circ \Phi)(x) = 0$ for $x < 1/b$.

Since $|f| \leq c$ for some c on $(1, \infty)$, we learn that on $(0, 1)$ that $|f \circ \Phi| \leq c$, and $|(f \circ \Phi)/x^2| \leq b^2 c$.

Since f is continuous a.e., so is $f \circ \Phi$. So the LHS is Riemann integrable too.