# MATH 3210 FINAL, FALL 2015

1. Let $f : M_n \to M_n$ take an $n \times n$ real matrix to its square, $f(A) = A^2$.

a [5]. What is the derivative of this map?

*Answer.* $Df_A(H) = \lim_{\varepsilon\to 0} \frac{(A+\varepsilon H)^2 - A^2}{\varepsilon} = AH + HA$.

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b [5]. Let $m_{ij} : M_n \to \mathbb{R}$ be the "what's the $(i, j)$-entry?" map. Compute $f^*(dm_{11} \wedge dm_{12})$.

*Answer.* First,

$$f^*(dm_{11} \wedge dm_{12}) = f^*(dm_{11}) \wedge f^*(dm_{12}) = d(f^*m_{11}) \wedge d(f^*m_{12}).$$

Now use $f^*(m_{ij}) = (M^2)_{ij} = \sum_k m_{ik}m_{kj}$. So

$$df^*m_{ij} = \sum_k d(m_{ik}m_{kj}) = \sum_k (dm_{ik}m_{kj} + m_{ik}dm_{kj})$$

and finally

$$\begin{aligned} d(f^*m_{11}) \wedge d(f^*m_{12}) &= \sum_k (dm_{1k}m_{k1} + m_{1k}dm_{k1}) \wedge \sum_l (dm_{1l}m_{l2} + m_{1l}dm_{l2}) \\ &= \sum_{k,l} (m_{k1}m_{l2}dm_{1k} \wedge dm_{1l} + m_{k1}m_{1l}dm_{1k} \wedge dm_{l2} \\ &\quad + m_{1k}m_{l2}dm_{k1} \wedge dm_{1l} + m_{1k}m_{1l}dm_{l2} \wedge dm_{k1}) \end{aligned}$$

(If you let $dM$ be the matrix full of 1-forms $dm_{ij}$, then $df^*m_{ij} = (dM\, M + M\, dM)_{ij}$, if you like that any better.)

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c [5]. Let $D$ be a diagonal matrix with $f(D) = I$.
An oracle tells you "the set of points in $f^{-1}(I)$ connected to $D$ is a manifold $X$".
What is the dimension of this $X$? (Your answer should be able to handle any choice of $D$.)

*Answer.* Its dimension is $\dim T_D f^{-1}(I)$, and $T_D f^{-1}(I)$ is the kernel of $Df|_D$,
which is $\{H : DH + HD = 0\}$.

Our $D$ is a diagonal matrix with some $+1$s and some $-1$s, say $k$ and $n-k$ of them; call this $\pm 1$ vector $\vec{d}$. Unwrapping the equation $HD + DH = 0$, we learn $H_{ij} = 0$ unless $d_i = -d_j$. (In that case $H_{ij}$ free, i.e. has no condition.) So the number of free entries in $H$ is $2k(n-k)$.

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2. Let $A, B$ be two manifolds-with-boundary.

a [5]. Give an example where $A \times B$ is not a manifold-with-boundary. (You don't have to prove it isn't one. Just be right!)

*Answer.* If both have boundary, e.g. $\mathbb{R}_{\geq 0}$, then $A \times B$ will have a corner.

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b [5]. Give an example where $A \times B$ *is* a manifold-with-boundary.

*Answer.* If $A$ has no boundary. *Really* extreme case: take $A$ empty.

3. Let $M = [\vec{c}_1 \cdots \vec{c}_n]$ be a square real matrix with columns $(\vec{c}_i)$.

a [5]. If the $(\vec{c}_i)$ form a basis, what are the possibilities for $\det M$?

*Answer.* It must be invertible, so $\det \neq 0$.

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b [5]. Assume they do form a basis, and furthermore, that the dual basis (with respect to usual dot product) is again $\vec{c}_1, \ldots, \vec{c}_n$. Now what are the possibilities for $\det M$?

*Answer.* $M$'s columns are orthonormal, i.e. $M^T M = I$. Then $\det M = \pm 1$.

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4 [10]. Let $\phi : V \twoheadrightarrow W$ be a linear map that's onto.
Say we have orientations $\mathcal{O}_V, \mathcal{O}_W$ of $V, W$. Use them to define an orientation of $\ker \phi$. Your answer should have the property that it flips if we flip the orientation of $V$ or of $W$.

*Answer.* Pick a basis $(v_i)$ for $\ker \phi$. Extend to a basis for $V$, by adding extra elements $(w_j)$. Then their images $(\phi(w_j))$ in $W$ will be a basis. Let

$$\mathcal{O}_{\ker \phi}(w_1, \ldots, w_{\dim W}) := \mathcal{O}_V(v_1, \ldots, v_{\dim \ker(\phi)}, w_1, \ldots, w_{\dim W}) / \mathcal{O}_W(\phi(w_1), \ldots, \phi(w_{\dim W})).$$

*Alternate answer.* First we show that if $A$ is oriented, we can use that to orient $A^*$, as follows; given a basis for $A^*$, call it positive or negative depending on whether the dual basis is positive or negative for $A$.

Then dualize $0 \to \ker \phi \hookrightarrow V \twoheadrightarrow W \to 0$ to get $0 \leftarrow V^*/W^* \leftarrow V^* \leftarrow W^* \to 0$.
On homework we already oriented $V^*/W^*$, whose dual we identify with $\ker \phi$.

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5. Let $M = U \cup V$ be a manifold, with $U, V$ open subsets.
Assume that the cohomology groups of $M, U, V, U \cap V$ are all finite-dimensional, so we can define the Poincaré polynomials of everybody, e.g. $p_M(t) = \sum_i (\dim H^i(M)) t^i$.

[10] Show $p_M = p_U + p_V$ iff $p_M(0) = p_U(0) + p_V(0)$.

*Answer.* $\Longrightarrow$ is trivial. For the reverse, look at the first row of Mayer-Vietoris:

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V)$$

By $p_M(0) = p_U(0) + p_V(0)$, the $1:1$ map $H^0(M) \to H^0(U) \oplus H^0(V)$ is onto. Hence the map $H^0(U) \oplus H^0(V) \to H^0(U \cap V)$ is zero.

I.e. the restriction maps $H^0(U) \to H^0(U \cap V)$ and $H^0(V) \to H^0(U \cap V)$ are zero.

But this row is about restrictions of locally constant functions. The function 1 on $U$ should restrict to 1 on $U \cap V$, and not be zero, unless $U \cap V$ is empty. Once you know $U \cap V = \emptyset$, then $p_M = p_U + p_V$ by Mayer-Vietoris.

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6 [10]. Let $M = \mathbb{R}^2 \setminus \{(0,0), (1,0)\}$. Use Mayer-Vietoris to compute $H^\bullet(M)$. Use whatever you want from calculations we've done before (but be explicit about what you're using).

*Answer.* Let $U = M \cap \{(x,y) : x < 1\}$, $V = M \cap \{(x,y) : x > 0\}$, so their intersection $U \cap V$ is $(0,1) \times \mathbb{R}$. We've already computed $H^*(U), H^*(V)$ (each the plane minus a single point), and $H^*(U \cap V)$ vanishes enough to not give us trouble with boundary maps:

$$\begin{aligned} 0 \ \to\ & H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \\ \to\ & H^1(M) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \\ \to\ & H^2(M) \to H^2(U) \oplus H^2(V) \to H^2(U \cap V) \to 0 \end{aligned}$$

is

$$\begin{aligned} 0 &\to H^0(M) \to \mathbb{R}^2 \to \mathbb{R} \\ &\to H^1(M) \to \mathbb{R}^2 \to 0 \\ &\to H^2(M) \to 0 \to 0 \to 0 \end{aligned}$$

and $M$ is connected, so $H^0(M) \cong \mathbb{R}$. Hence the first row is short exact, $H^1(M) \cong \mathbb{R}^2$, $H^2(M) = 0$.

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7 [10]. Let $A < B < C$ be three vector spaces. Define a natural map $C/A \twoheadrightarrow C/B$ and compute its kernel.

*Answer.* $\vec{c} + A \mapsto \vec{c} + B$. The kernel is $\{\vec{c} + A\ : \vec{c} \in B\} = B/A$.

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8. Let $\alpha = x\ dy + y\ dz$ on $\mathbb{R}^3$.

[5]. Compute $\alpha, \quad \alpha \wedge d\alpha, \quad \alpha \wedge d\alpha \wedge d^2\alpha, \quad \alpha \wedge d\alpha \wedge d^2\alpha \wedge d^3\alpha, \quad \ldots$

*Answer.*

$$d\alpha = dx \wedge dy + dy \wedge dz$$

$$\begin{aligned} \alpha \wedge d\alpha &= (x\ dy + y\ dx) \wedge (dx \wedge dy + dy \wedge dz) \\ &= x\ dy \wedge dx \wedge dy + x\ dy \wedge\ dy \wedge dz \\ &\quad + y\ dx \wedge dx \wedge dy + y\ dx \wedge\ dy \wedge dz \\ &= 0 + 0 + 0 + y\ dx \wedge\ dy \wedge dz \end{aligned}$$

After that, everything has a $d^2$ in it, so is zero.

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9. Let $f : U \to M$ be a smooth function, and $\alpha \in \Omega^k(M)$. For each of the following, prove or give a counterexample.

a [5]. If $\alpha$ is closed, is $f^*\alpha$ closed?

*Answer.* Yes, $d(f^*\alpha) = f^*(d\alpha) = f^*(0) = 0$.

b [5]. If $f^*\alpha$ is closed, is $\alpha$ closed?

*Answer.* No, consider $f : \{0\} \to \mathbb{R}^1$, and let $\alpha = x \in \Omega^0(M)$, or any other non-closed form.

c [5]. If $\alpha$ is exact, is $f^*\alpha$ exact?

*Answer.* Yes, $\alpha = d\beta$ implies $f^*(\alpha) = f^*(d\beta) = df^*(\beta)$.

d [5]. If $f^*\alpha$ is exact, is $\alpha$ exact?

*Answer.* No, same as in (b).