1. Let \( f : M_n \to M_n \) take an \( n \times n \) real matrix to its square, \( f(A) = A^2 \).

a [5]. What is the derivative of this map?

Answer. \( Df_A(H) = \lim_{\varepsilon \to 0} \frac{(A + \varepsilon H)^2 - A^2}{\varepsilon} = AH + HA \).

b [5]. Let \( m_{ij} : M_n \to \mathbb{R} \) be the “what’s the \((i, j)\)-entry?” map. Compute \( f^*(dm_{11} \wedge dm_{12}) \).

Answer. First, \( f^*(dm_{11} \wedge dm_{12}) = f^*(dm_{11}) \wedge f^*(dm_{12}) = d(f^*m_{11}) \wedge d(f^*m_{12}) \).

Now use \( f^*(m_{ij}) = (M^2)_{ij} = \sum_k m_{ik} m_{kj} \). So

\[
 df^*m_{ij} = \sum_k d(m_{ik} m_{kj}) = \sum_k (dm_{ik} m_{kj} + m_{ik} dm_{kj})
\]

and finally

\[
 d(f^*m_{11}) \wedge d(f^*m_{12}) = \sum_k (dm_{ik} m_{k1} + m_{ik} dm_{k1}) \wedge \sum_l (dm_{il} m_{l1} + m_{il} dm_{l1})
\]

\[
 = \sum_{k, l} (m_{k1} m_{l2} dm_{ik} \wedge dm_{l1} + m_{k1} m_{l1} dm_{ik} \wedge dm_{l2} + m_{1k} m_{l2} dm_{k1} \wedge dm_{l1} + m_{1k} m_{l1} dm_{k2} \wedge dm_{k1})
\]

(If you let \( dM \) be the matrix full of 1-forms \( dm_{ij} \), then \( df^*m_{ij} = (dM M + M dM)_{ij} \), if you like that any better.)

c [5]. Let \( D \) be a diagonal matrix with \( f(D) = I \).

An oracle tells you “the set of points in \( f^{-1}(I) \) connected to \( D \) is a manifold \( X \)”.

What is the dimension of this \( X \)? (Your answer should be able to handle any choice of \( D \).)

Answer. Its dimension is \( \dim T_D f^{-1}(I) \), and \( T_D f^{-1}(I) \) is the kernel of \( Df|_D \), which is \( \{H : DH + HD = 0\} \).

Our \( D \) is a diagonal matrix with some +1s and some −1s, say \( k \) and \( n - k \) of them; call this ±1 vector \( \bar{d} \). Unwrapping the equation \( HD + DH = 0 \), we learn \( H_{ij} = 0 \) unless \( d_i = -d_j \).

(In that case \( H_{ij} \) free, i.e. has no condition.) So the number of free entries in \( H \) is \( 2k(n-k) \).

2. Let \( A, B \) be two manifolds-with-boundary.

a [5]. Give an example where \( A \times B \) is not a manifold-with-boundary. (You don’t have to prove it isn’t one. Just be right!)

Answer. If both have boundary, e.g. \( \mathbb{R}_{\geq 0} \), then \( A \times B \) will have a corner.

b [5]. Give an example where \( A \times B \) is a manifold-with-boundary.

Answer. If \( A \) has no boundary. Really extreme case: take \( A \) empty.
3. Let \( M = [c_1 \cdots c_n] \) be a square real matrix with columns \((c_i)\).

a [5]. If the \((c_i)\) form a basis, what are the possibilities for \( \det M \)?

\textit{Answer.} It must be invertible, so \( \det \neq 0 \).

b [5]. Assume they do form a basis, and furthermore, that the dual basis (with respect to usual dot product) is again \( c_1, \ldots, c_n \). Now what are the possibilities for \( \det M \)?

\textit{Answer.} \( M \)'s columns are orthonormal, i.e. \( M^TM = I \). Then \( \det M = \pm 1 \).

4 [10]. Let \( \phi : V \to W \) be a linear map that’s onto.

Say we have orientations \( \mathcal{O}_V, \mathcal{O}_W \) of \( V, W \). Use them to define an orientation of \( \ker \phi \).

\textit{Your answer should have the property that it flips if we flip the orientation of \( V \) or of \( W \).}

\textit{Answer.} Pick a basis \((v_i)\) for \( \ker \phi \). Extend to a basis for \( V \), by adding extra elements \((w_i)\).

Then their images \((\phi(w_i))\) in \( W \) will be a basis. Let

\[ \mathcal{O}_{\ker \phi}(w_1, \ldots, w_{\dim W}) := \mathcal{O}_V(v_1, \ldots, v_{\dim \ker(\phi)}, w_1, \ldots, w_{\dim W}) / \mathcal{O}_W(\phi(w_1), \ldots, \phi(w_{\dim W})). \]

\textit{Alternate answer.} First we show that if \( A \) is oriented, we can use that to orient \( A^* \), as follows; given a basis for \( A^* \), call it positive or negative depending on whether the dual basis is positive or negative for \( A \).

Then dualize \( 0 \to \ker \phi \to V \to W \to 0 \) to get \( 0 \to V^*/W^* \to V^* \to W^* \to 0 \).

On homework we already oriented \( V^*/W^* \), whose dual we identify with \( \ker \phi \).

5. Let \( M = U \cup V \) be a manifold, with \( U, V \) open subsets.

Assume that the cohomology groups of \( M, U, V, U \cap V \) are all finite-dimensional, so we can define the Poincaré polynomials of everybody, e.g. \( p_M(t) = \sum_i (\dim H^i(M)) t^i \).

[10] Show \( p_M = p_U + p_V \) iff \( p_M(0) = p_U(0) + p_V(0) \).

\textit{Answer.} \( \implies \) is trivial. For the reverse, look at the first row of Mayer-Vietoris:

\[ 0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \]

By \( p_M(0) = p_U(0) + p_V(0) \), the 1 : 1 map \( H^0(M) \to H^0(U) \oplus H^0(V) \) is onto. Hence the map \( H^0(U) \oplus H^0(V) \to H^0(U \cap V) \) is zero.

I.e. the restriction maps \( H^0(U) \to H^0(U \cap V) \) and \( H^0(V) \to H^0(U \cap V) \) are zero.

But this row is about restrictions of locally constant functions. The function 1 on \( U \) should restrict to 1 on \( U \cap V \), and not be zero, unless \( U \cap V \) is empty. Once you know \( U \cap V = \emptyset \), then \( p_M = p_U + p_V \) by Mayer-Vietoris.

6 [10]. Let \( M = \mathbb{R}^2 \setminus \{(0,0), (1,0)\} \). Use Mayer-Vietoris to compute \( H^*(M) \). Use whatever you want from calculations we’ve done before (but be explicit about what you’re using).

\textit{Answer.} Let \( U = M \cap \{(x,y) : x < 1\}, V = M \cap \{(x,y) : x > 0\} \), so their intersection \( U \cap V \) is \( (0,1) \times \mathbb{R} \). We’ve already computed \( H^*(U), H^*(V) \) (each the plane minus a single point), and \( H^*(U \cap V) \) vanishes enough to not give us trouble with boundary maps:

\[
\begin{align*}
0 & \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \\
& \to H^1(M) \to H^1(U) \oplus H^1(V) \to H^1(U \cap V) \\
& \to H^2(M) \to H^2(U) \oplus H^2(V) \to H^2(U \cap V) \to 0
\end{align*}
\]
is

\[ 0 \rightarrow H^0(M) \rightarrow \mathbb{R}^2 \rightarrow \mathbb{R} \]
\[ \rightarrow H^1(M) \rightarrow \mathbb{R}^2 \rightarrow 0 \]
\[ \rightarrow H^2(M) \rightarrow 0 \rightarrow 0 \rightarrow 0 \]

and \( M \) is connected, so \( H^0(M) \cong \mathbb{R} \). Hence the first row is short exact, \( H^1(M) \cong \mathbb{R}^2 \), \( H^2(M) = 0 \).

7 [10]. Let \( A < B < C \) be three vector spaces. Define a natural map \( C/A \rightarrow C/B \) and compute its kernel.

\textit{Answer.} \( \overline{c} + A \mapsto \overline{c} + B \). The kernel is \( \{ \overline{c} + A : \overline{c} \in B \} = B/A \).

8. Let \( \alpha = x \, dy + y \, dz \) on \( \mathbb{R}^3 \).

\[ \text{[5]. Compute } \alpha, \quad \alpha \wedge d\alpha, \quad \alpha \wedge d\alpha \wedge d^2\alpha, \quad \alpha \wedge d\alpha \wedge d^2\alpha \wedge d^3\alpha, \quad \ldots \]

\textit{Answer.}

\[ d\alpha = dx \wedge dy + dy \wedge dz \]
\[ \alpha \wedge d\alpha = (x \, dy + y \, dx) \wedge (dx \wedge dy + dy \wedge dz) \]
\[ = x \, dy \wedge dx \wedge dy + x \, dy \wedge dy \wedge dz \]
\[ + y \, dx \wedge dx \wedge dy + y \, dx \wedge dy \wedge dz \]
\[ = 0 + 0 + y \, dx \wedge dy \wedge dz \]

After that, everything has a \( d^2 \) in it, so is zero.

9. Let \( f : U \rightarrow M \) be a smooth function, and \( \alpha \in \Omega^k(M) \). For each of the following, prove or give a counterexample.

a [5]. If \( \alpha \) is closed, is \( f^*\alpha \) closed?

\textit{Answer.} Yes, \( d(f^*\alpha) = f^*(d\alpha) = f^*(0) = 0 \).

b [5]. If \( f^*\alpha \) is closed, is \( \alpha \) closed?

\textit{Answer.} No, consider \( f : \{0\} \rightarrow \mathbb{R}^1 \), and let \( \alpha = x \in \Omega^0(M) \), or any other non-closed form.

c [5]. If \( \alpha \) is exact, is \( f^*\alpha \) exact?

\textit{Answer.} Yes, \( \alpha = d\beta \) implies \( f^*(\alpha) = f^*(d\beta) = df^*(\beta) \).

d [5]. If \( f^*\alpha \) is exact, is \( \alpha \) exact?

\textit{Answer.} No, same as in (b).