MATH 3210 MIDTERM #1, FALL 2015

Name, **written slowly and legibly:** ________________________________

In each answer, write as much (on front and back) as it takes to convey your thought process; full English sentences are much easier to give credit to than bare, unmotivated scribbled formulæ. (They won't do any good if they can't be read, so *do* put effort into making them legible.)

Feel free to ask me questions during the test, especially if you need a little reminder about a definition. Worst case is I don't answer. (It's very sad to afterward hear "I didn't realize I could ask you that" — find out!)

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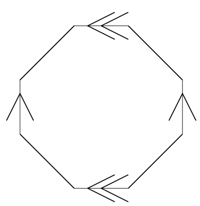
1. What are the following surfaces? Possible answers: "a Möbius strip", "a pair of pants", "a Klein bottle", "a doughnut with 3 holes", etc.

a [10].



Answer. A torus with a hole (whose boundary is a circle, made from the four unpaired edges).

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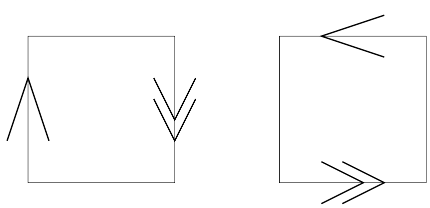
b [10].



Answer. A cylinder (not a Möbius strip). The two flips add up to no flip.

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2. Let $\det : M_2(\mathbb{R}) \to \mathbb{R}$ be the determinant function, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$.

a [30]. What are the critical points (i.e. those $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ at which $D\det$ is not onto) and critical values (the images of the critical points)?

Answer. We compute the derivative, by taking the partial derivatives with respect to the four coordinates, obtaining $[d \; -c \; -b \; a]$. The rank of this matrix is 1 unless $a = b = c = d = 0$, so the only critical point is the zero matrix. Its image is 0, the only critical value.

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b [20]. What's the dimension of $\det^{-1}(10)$, and how did you compute it?

Answer. We proved in class that if $f : A \to B$ has a regular value p, then $\dim f^{-1}(p) = \dim A - \dim B$; in this case that's $4 - 1 = 3$.

3a [20]. Let $f : M \to N$ be a smooth map. Define

$$g : M \to M \times N, \qquad m \mapsto (m, f(m))$$
$$p : M \times N \to M, \qquad (m, n) \mapsto n$$

and use them to show that f is a composite of an immersion (show every Dg is $1:1$) and a submersion (show every Dp is onto).

Answer. First check the composite statement: $(p \circ g)(m) = p(g(m)) = p(m, f(m)) = f(m)$. So $f = p \circ g$.

Now compute the derivative $Dg : T_m M \to T_{f(m)}(M \times N)$ of $m \mapsto (m, f(m))$ A small change $\vec{v}$ in m gives the same small change $\vec{v}$ in m, but a different change $Df_m(\vec{v})$ in $f(m)$, so the derivative is

$$Dg_m(\vec{v}) = (\vec{v}, Df_m(\vec{v}))$$

and this RHS is only $\vec{0}$ if $\vec{v} = \vec{0}$. So Dg_m is $1:1$.

The derivative of p is even easier:

$$Dp_{(m,n)} : T_{(m,n)}(M \times N) \to T_n N, \qquad (\vec{v}, \vec{w}) \mapsto \vec{w}$$

so in particular, for any $\vec{w} \in T_n N$ we have $Dp_{(m,n)}(\vec{0}, \vec{w}) = \vec{w}$. So $Dp_{(m,n)}$ is onto.

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b [10]. If $f : V \to W$ is a linear map, show it's a composite of a submersion and an immersion (the opposite of part (a)).

Answer. Since this is the opposite of part (a), we want the first map to be onto, and the second to be $1:1$. Write f as the composite

$$V \twoheadrightarrow \mathit{image}(f) \hookrightarrow W$$

where the two maps are $\vec{v} \mapsto f(\vec{v})$ and the inclusion map $\vec{w} \mapsto \vec{w}$. These are linear, and onto and $1:1$ respectively, so their derivatives are also.