MATH 3210 MIDTERM #2, FALL 2015

1. Let $f : \mathbb{R} \to \mathbb{R}^2$ take $m \mapsto (\cos m, \sin m)$.

a. Let x denote the first coordinate function on $\mathbb{R}^2$.
Compute $f^*(x)$ [10 points] and $f^*(dx)$ [15 points].

Answer. $f^*(x) = \cos m$, $f^*(dx) = df^*(x) = d\cos m = -\sin m\, dm$

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b [10]. Prove that $\alpha = m^2\, dm$ is *not* of the form $f^*(\beta)$, for β any 1-form on $\mathbb{R}^2$. (The f is from part (a), of course.)

Answer. Since f is 2π-periodic, so too must $f^*(\beta)$ be, but $m^2\, dm$ isn't.

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2. Let $\Omega^k_c(\mathbb{R}^n)$ denote the forms of **bounded support**, i.e. for each such form α there exists a radius r such that $\alpha = 0$ outside the ball of radius r around $\vec{0}$.

a [10]. Show that $d(\Omega^k_c(\mathbb{R}^n)) \leq \Omega^{k+1}_c(\mathbb{R}^n)$.

Answer. If α is zero on some region (i.e. outside some ball), then $d\alpha$ is too.

(It's not true that if α is zero at a *point,* then $d\alpha$ is zero there too, since $d\alpha$ uses partial derivatives of α.)

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b [15]. Define $H^i_c(\mathbb{R}^N)$ as $\ker(\Omega^i_c \xrightarrow{d} \Omega^{i+1}_c) / \mathrm{image}(\Omega^{i-1}_c \xrightarrow{d} \Omega^i_c)$.

Compute, in detail, the dimension of $H^0_c(\mathbb{R})$.

(We did this *without* the $_c$ before. I am *not* asking you to regurgitate that calculation.)

Answer. The image is coming from Ω^{-1}_c which was defined to be 0. The kernel is functions f with $df = 0$, i.e. constant functions, that are required to be 0 outside some interval. So the constant is 0, i.e. the kernel is just the zero function.

Hence $H^i_c(\mathbb{R}) = 0$.

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c [20]. Compute, in detail, the dimension of $H^1_c(\mathbb{R})$.

Answer. As without the $_c$, the only 2-form is 0, so the kernel is all compactly supported 1-forms. Now, when is $\alpha = df$, both zero outside some ball $(-r, r)$?

Since $f(-r) = 0$, we can compute f by $f(x) = \int_{t=-r}^{x} \alpha$ or even $\int_{t=-\infty}^{x} \alpha$. But now we want the condition $f(\infty) = 0$, i.e. $\int_{\mathbb{R}} \alpha = 0$. Put another way, the image of $d : \Omega^0_c(\mathbb{R}) \to \Omega^1_c(\mathbb{R})$ is the kernel of $\int_{\mathbb{R}} : \Omega^1_c(\mathbb{R}) \twoheadrightarrow \mathbb{R}$, hence the quotient by that image is 1-dimensional.

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3 [20]. Say $a_1, \ldots, a_n \in V$ is a basis, with dual basis $b_1, \ldots, b_n \in V^*$.

No wait, I changed my mind on one vector; make the first basis be $a_1, \ldots, a_{n-1}, a'_n$.

How does $b_1, \ldots, b_n$ change? Shouldn't be a lot, right? Describe the new dual basis $b'_1, \ldots, b'_n$ as precisely as you can (i.e. write it in terms of the old dual basis).

Reminder (or "hint", except that you should **not need** hints like these): consider the easiest possible case of $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ first, to gain intuition for the general answer.

Answer. The easiest case is (a_i) the standard basis of column vectors, (b_i) the standard basis of row vectors. If we assemble them into matrices A, B then the dual-basis condition is $BA = 1$ (which is stupid because $A, B = 1$ in this easiest case).

If we change a_n to $a'_n = \sum_{i=1}^n \lambda_i a_i$, the new matrix $A' = [a_1 a_2 \cdots a_{n-1} a'_n]$ is the identity outside the last column, so its inverse B' is also the identity outside the last column. That is, our new dual basis (b'_i) will have $b'_i = b_i + \mu_i b_n$, for μ_i the entries in the inverse matrix.

So, let's try that for the general case, and find out what the μ_i are. We want

$$\begin{aligned} b'_i(a_j) &= \delta_{ij} \qquad \text{for } j < n \\ b'_i(a'_n) &= \delta_{in} \end{aligned}$$

For the first,

$$b'_i(a_j) = (b_i + \mu_i b_n)(a_j) = b_i(a_j) + \mu_i b_n(a_j) = \delta_{ij} + \mu_i 0 = \delta_{ij}$$

and the second,

$$b'_i(a'_n) = (b_i + \mu_i b_n)\Big(\sum_j \lambda_j a_j\Big) = \lambda_i + \mu_i \lambda_n$$

so $\mu_i = (\delta_{in} - \lambda_i)/\lambda_n$, which is what you got if you computed the inverse matrix.

(Why is $\lambda_n \neq 0$? If it were zero, then a'_n would be in the span of $a_1, \ldots, a_{n-1}$, so the new "basis" wouldn't actually be a basis.)