NOTES ON KASHIWARA-SCHAPIRA

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Goals:

- understand $\mathcal{D}$-modules well enough to prove Beilinson-Bernstein, say from
  - http://www.math.harvard.edu/~gaitsgde/grad_2009/Ginzburg.pdf
  - http://yisun.mit.edu/notes/dmod.pdf
  - https://www.math.utah.edu/~milicic/Eprints/penrose.pdf
- understand perverse sheaves well enough to define the maps in the Riemann-Hilbert correspondence, and the quivers in
  - https://arxiv.org/abs/math/9907152 Perverse sheaves on Grassmannians, Tom Braden
  - https://projecteuclid.org/euclid.dmj/1077229136 Perverse sheaves on rank stratifications, Tom Braden and Mikhail Grinberg
- understand when Lagrangians are smooth enough to define maps on what sort of homology, e.g. the Springer representations
- Euler obstructions and the correspondence between constructible functions and Lagrangian cycles
- functorial operations on the derived category of perverse sheaves, and how they decategorify to operations on functions
- dream: connect the theory of CSM classes with that of Deligne's weight filtration on H^*, maybe via Schürmann https://arxiv.org/abs/math/0511175
- understand the appearance of CSM classes in http://front.math.ucdavis.edu/1305.7462
- mixed Hodge modules http://front.math.ucdavis.edu/1307.5152
- microlocal sheaves:
  - https://arxiv.org/abs/math/0509440 Microlocal Perverse Sheaves, S. Gelfand, R. MacPherson, K. Vilonen
  - https://arxiv.org/abs/1506.07050 Microlocal sheaves and quiver varieties, Roman Bezrukavnikov, Mikhail Kapranov
  - https://arxiv.org/abs/1512.08942 Cluster varieties from Legendrian knots, Vivek Shende, David Treumann, Harold Williams, Eric Zaslow
  - https://arxiv.org/abs/math/9906200 Ind-Sheaves, distributions, and microlocalization, Masaki Kashiwara, Pierre Schapira
  - https://arxiv.org/abs/1511.08961 Microlocal Category, Dmitry Tamarkin (152 pp!)
  - http://arxiv.org/abs/1503.06240 Categories of (co)isotropic linear relations, Alan Weinstein

Date: August 31, 2016.

1. MONDAY 8/29

1.1. **Noncommutative algebra.** Def: $U\mathfrak{g}$. A filtered, noncommutative algebra A. Its associated graded is **Poisson**.[1] Poisson manifolds (locally) have **symplectic leaves**, on which $\mathrm{gr}\ Z(A)$ acts as scalars. $Z(U\mathfrak{g}) = (U\mathfrak{g})^G \cong (\mathrm{Sym}\ \mathfrak{g})^G \cong (\mathrm{Sym}\ \mathfrak{t})^W$. The most interesting fiber is $\mathcal{N}$, the **nilpotent cone**. Let

$$(U\mathfrak{g})_0 := U\mathfrak{g}/\langle Z(U\mathfrak{g}) \text{ acts as it does on the trivial rep}\rangle$$

called **trivial central character**.

Def: $\mathcal{D}_M$, for M a smooth variety. Its associated graded is $\mathcal{O}_{T^*M}$, symplectic.

Def: $\mathcal{D}$-module.

Ex: Various distributions on the line (at/away from a point)

2. WEDNESDAY 8/31

Def: **good filtration** of a $\mathcal{D}$-module M is compatible with the filtration on $\mathcal{D}$, and $\mathrm{gr}\ M$ is coherent in each degree.

Def: coisotropic (= involutive).

Thms to come:

- If M is f.g., it has a good filtration.
- The **singular support** of $\mathrm{gr}\ M$ on T^*M is coisotropic [Gabber81, building on Sato-Kawai-Kashiwara73]. It doesn't depend on the choice of filtration, and comes with well-defined multiplicities, defining the **characteristic cycle** of M [Bernstein, p5-7 in Ginzburg].

Def: **holonomic** $\mathcal{D}$-modules have **Lagrangian** support. This property will be preserved by various operations we will define on $\mathcal{D}$-modules.

2.1. **Lagrangians.** Def. **conormal bundle, conormal variety** CY. Linear subspaces as examples.

Lemma 2.1 (Arnol'd). *Let $X \subseteq T^*M$ be closed, reduced, irreducible, Lagrangian (at its smooth points), and conical.*

1. *If we identify M with the zero section $\subseteq T^*M$, then $X \cap M = \pi(X)$, where $\pi : T^*M \to M$ is the projection. Hence this set $Y \subseteq M$ is closed and irreducible.*
2. *X is the conormal variety of Y.*

Proof.

1. $\pi(X) \supseteq \pi(X \cap M) = X \cap M$. For the converse, let $m \in \pi(X)$ so $\exists (m, \vec{v}) \in X$. Then by conicality, each $(m, z\vec{v}) \in X$, so taking $z \to 0$ and using closedness $(m, \vec{0}) \in X$. Hence $m \in X \cap M$.
2. Since Y_{reg} is open dense in Y, $X^\circ := \pi^{-1}(Y_{reg}) \cap X$ is open dense in X. Since X° is isotropic, it's contained in CY_{reg}, but then dense in it by dimension count. Hence their closures X and CY agree.

∎

[1] Specifically, let $a \in \mathrm{gr}_i\ A$, $b \in \mathrm{gr}_j\ A$, with lifts $\bar{a} \in A_i, \bar{b} \in A_j$. Then $\bar{a}\bar{b} - \bar{b}\bar{a} \in A_{i+j}$ a priori, but is actually in A_{i+j-1} by the commutativity of $\mathrm{gr}\ A$, so we define $\{a, b\} \in \mathrm{gr}_{i+j-1}\ A$ as the coset of $\bar{a}\bar{b} - \bar{b}\bar{a}$. Checking well-definedness and Poissonness is routine.

Application of the lemma: projective duality[2]. This will be related to a Fourier transform for $\mathcal{D}$-modules.

One *could* linearly extend this correspondence to one between conical Lagrangian cycles and constructible functions on M, but we'll prefer a different correspondence.

If $\mathfrak{g}$ acts on M, i.e. $\mathfrak{g} \to \mathrm{Vec}(M)$, then $U\mathfrak{g} \to H^0(\mathcal{D}_M)$. Example: G acts on G/B.

Thm to come: (Beilinson-Bernstein)

- $U\mathfrak{g} \to H^0(\mathcal{D}_{G/B})$ factors through $(U\mathfrak{g})_0$ and is then an isomorphism.
- $H^{i>0}(\mathcal{D}_{G/B}) = 0$.
- There is an equivalence of categories, of $(U\mathfrak{g})_0$-modules and $\mathcal{D}_{G/B}$-modules.

(We'll twist this to get other central characters later.)

Apparently one's supposed to think of this as giving two kinds of resolutions of the nilpotent cone: one commutative (but no longer affine) to T^*G/B, the other affine (but no longer commutative) to $(U\mathfrak{g})_0$. And then the interesting bit is that they give the same category.

So we go from rep theory (at trivial central character) to $\mathcal{D}_{G/B}$-modules, and then would like to go to T^*G/B.

Proof of SKK/Gabber, due to Knop, via Ginzburg. This is true more generally for filtered A with $\mathrm{gr}\, A$ commutative and hence Poisson. Let $I_M \leq \mathrm{gr}\, A$ be the support of $\mathrm{gr}\, M$; then the coisotropic condition is that $\{I_M, I_M\} \leq I_M$.

... p9-10 of Ginzburg □

[2]Tons more on this at `http://arxiv.org/abs/math/0112028`