CONSTRUCTING LIE ALGEBRAS FROM DYNKIN DIAGRAMS

ALLEN KNUTSON

There are four classes of mathematical objects to connect up here:

- complex semisimple Lie groups (complexifications of compact with discrete center),
- centerless complex Lie algebras,
- root systems, and
- Dynkin diagrams.

Let us review the connections.

From a Lie group we get a Lie algebra.

From a compact Lie group and a chosen maximal torus in the group, we get a root system. (One can get by with just the Lie algebra, but we won't do this.) Since all tori in the compact group are conjugate, the root system is well defined up to isomorphism. In fact, the space of tori in K is $K/N(T)$ and has fundamental group the Weyl group W, so the root system is well defined up to the action of W. (Not every automorphism of a root system comes from the Weyl group.)

Given a root system, we can choose a positive system, and from its simple roots construct the Dynkin diagram. The positive systems are all conjugate to one another via W, so the Dynkin diagram is well-defined.

To go in reverse: given a Dynkin diagram, we can reconstruct the set of positive roots (as intermediate positions in the find-the-highest-root game), so root systems exactly correspond to Dynkin diagrams.

Given one connected group for an algebra, we know how to construct all the others. (It turns out that for the Lie algebras here, the connected component of the automorphism group of the Lie algebra fits the bill.)

It remains to construct, from a given root system, a Lie algebra having it as root system. The fundamental obstruction is that this process is *not* functorial – isomorphisms of root systems do not induce in a natural way isomorphisms of Lie algebras. This is because of a simple fact: the short exact sequence $T \to N(T) \to W$ defining the Weyl group does not split, in general.

Exercise. Show that it doesn't split for $SU(2)$. How about $SO(n)$?

It will turn out that it *almost* splits – there is a subgroup $N_{\mathbb{Z}}(T)$ of $N(T)$ that maps onto W, where the kernel is a power of $O(1) = Z_2$ rather than $U(1)$. So we need to replace the root system with a sort of double cover acted on faithfully by $N_{\mathbb{Z}}(T)$, and build from there.

Another viewpoint is that while there shouldn't be a functorial connection between root systems and Lie algebras, we may ask for a functor from Dynkin diagrams directly to Lie algebras.

Recall that an invariant form on $\mathfrak{g}$ induces one on the root system, and that during the classification of Dynkin diagrams we found out that in any root system, either all the roots are the same length or there are only two lengths. (In fact the Weyl group acts either with one or two orbits.) If there is just one length we call the root system **simply-laced**. This is equivalent to the Dynkin diagram having no double or triple bonds; by the classification we know these are the ADE Dynkin diagrams $\{A_n\}, \{D_n\}, E_6, E_7, E_8$.

I learned the following construction from Richard Borcherds, who claims that everyone doing vertex operator algebras around 1990 basically knew it.

Let D be a simply-laced Dynkin diagram, with vertices $\alpha_1, \alpha_2, \ldots, \alpha_l$. Let L be the free abelian group on the $\{\alpha_i\}$. In this construction one should think of the α_i as the simple *coroots*, living in the Lie algebra of the torus, *not* its dual. So eventually $L \otimes \mathbb{R}$ will be $\mathfrak{t}$, not $\mathfrak{t}^*$.

This group L comes with a natural inner product $\langle , \rangle$: let each $\langle \alpha_i, \alpha_i \rangle = 2$, disconnected dots be orthogonal, and connected dots have inner product -1. Let Δ be defined as the set of vectors of norm square 2.

Exercise. (Hard?) Show that this Δ agrees with that given by the find-the-highest-root game.

Define a covering group $\hat{L}$ with generators

$$\exp(\alpha_1), \exp(\alpha_2), \ldots, \exp(\alpha_l), \tau$$

and relations

$$\tau^2 = 1,$$

$$\tau \text{ is central},$$

$$[\exp(\alpha_i), \exp(\alpha_j)] = \tau^{\#\text{ of bonds}} \qquad (\text{this \# is 0 or 1}).$$

Note that $\hat{L}/\langle \tau \rangle \cong L$. Let $\pi : \hat{L} \to L$ be the covering map, with kernel $\langle \tau \rangle$.

We now use this to define a Lie algebra (over $\mathbb{Z}$, just because we can – you may tensor with $\mathbb{C}$ if you like).

Let $\hat{\Delta}$ be the double cover inside $\hat{L}$ of the coroot system $\Delta \subset L$. Let M be the free abelian group on $\hat{\Delta}$, modulo the relations $\vec{\tau e} = -\vec{e}$. Finally, let $\mathfrak{g}$ be $L \oplus M$. At this point it is clear that $\mathfrak{g}$ has the right dimension – the rank of the torus plus the number of roots. Essentially, the generators $\{\vec{e}\}_{e \in \hat{\Delta}}$ are root vectors, where we have chosen *two* opposed generators of each root space.

We now have to define a bracket on $\mathfrak{g}$. Start with

$$[L, L] = 0.$$

For the rest, let $X \in L$, and $e, f \in \hat{\Delta}$.

$$[X, \vec{e}] = -[\vec{e}, X] = \langle X, \pi(e) \rangle \, \vec{e}$$

$$[\vec{e}, \vec{f}] = \vec{ef} \quad \text{if } ef \in \hat{\Delta}$$

The next two are the most subtle, dealing with the case that $\pi(e) + \pi(f) = 0$. In the Lie algebra, this comes from bracketing a raising operator with the corresponding lowering operator, and therefore landing in $\mathfrak{t}$.

$$[\vec{e}, \vec{f}] = \pi(e) \quad \text{if } ef = 1$$

$$[\vec{e}, \vec{f}] = -\pi(e) \quad \text{if } ef = \tau$$

Finally,

$$[\vec{e}, \vec{f}] = 0 \quad \text{if } \pi(e) + \pi(f) \text{ is neither } 0 \text{ nor in } \Delta.$$

We now have to check well-definedness (because we imposed relations), antisymmetry, and the Jacobi identity $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.

1.1. **Well-definedness.** We have to see if the relation $\vec{\tau e} = -\vec{e}$ is respected.

$$[X, \vec{\tau e}] = [X, -\vec{e}] = -[X, \vec{e}] = -\langle X, \pi(e)\rangle\, \vec{e} = \langle X, \pi(\tau e)\rangle\, \vec{\tau e} \quad \text{indeed}$$

$$[\vec{\tau e}, \vec{f}] = -[\vec{e}, \vec{f}] = -\vec{ef} = \vec{\tau ef} \quad \text{indeed}$$

If $ef = 1$, then $\tau ef = \tau$, so

$$[\vec{\tau e}, \vec{f}] = -[\vec{e}, \vec{f}] = -\pi(e) \quad \text{good}$$

and conversely if $ef = \tau$, then $\tau ef = 1$, so

$$[\vec{\tau e}, \vec{f}] = -[\vec{e}, \vec{f}] = \pi(e) \quad \text{also good.}$$

1.2. **Antisymmetry.** We need to check on $[L, L]$, on $[L, M]$ vs. $[M, L]$, and on $[M, M]$. The first two are trivial.

Exercise. Check that the rule for the bracket on $[M, M]$ is antisymmetric.

1.3. **The Jacobi identity.** This has many dull cases. By the cyclic symmetry, it is enough to check $[L, [L, L]]$, $[L, [L, M]]$, $[L, [M, M]]$, and $[M, [M, M]]$. Then of course the elements of M can be taken from the basis $\hat{\Delta}$. To get us started: any $[L, [L, L]] = 0$, and

$$[X, [Y, \vec{e}]] + [Y, [\vec{e}, X]] + [\vec{e}, [X, Y]] = [X, [Y, \vec{e}]] - [Y, [X, \vec{e}]] + 0$$

and the two terms on the RHS are both $\langle X, e\rangle\langle Y, e\rangle\vec{e}$, so they cancel.

For the $[L, [M, M]]$ case, we face the four cases

- $\pi(e) + \pi(f)$ a root,
- $\pi(e) + \pi(f)$ neither a root nor zero,
- $ef = 1$, and
- $ef = \tau$.

In fact the first two cases fold together, if we let $\vec{ef} \equiv 0$ for $\pi(e) + \pi(f)$ not a root, and give

$$\begin{aligned}
[X, [\vec{e}, \vec{f}]] + [\vec{e}, [\vec{f}, X]] + [\vec{f}, [X, \vec{e}]] &= [X, \vec{ef}] - [\vec{e}, \langle X, \pi(f)\rangle \vec{f}] + [\vec{f}, \langle X, \pi(e)\rangle \vec{e}] \\
&= \langle X, \pi(ef)\rangle \vec{ef} - \langle X, \pi(f)\rangle [\vec{e}, \vec{f}] + \langle X, \pi(e)\rangle [\vec{f}, \vec{e}] \\
&= \langle X, \pi(ef)\rangle \vec{ef} - (\langle X, \pi(f)\rangle + \langle X, \pi(e)\rangle)[\vec{e}, \vec{f}] \\
&= (\langle X, \pi(ef)\rangle - (\langle X, \pi(f)\rangle + \langle X, \pi(e)\rangle))\vec{ef} = 0
\end{aligned}$$

If $ef = 1$, we have

$$\begin{aligned}
[X, [\vec{e}, \vec{f}]] + [\vec{e}, [\vec{f}, X]] + [\vec{f}, [X, \vec{e}]] &= [X, \pi(e)] - [\vec{e}, \langle X, \pi(f)\rangle \vec{f}] + [\vec{f}, \langle X, \pi(e)\rangle \vec{e}] \\
&= 0 + \langle X, \pi(f) + \pi(e)\rangle [\vec{f}, \vec{e}] = 0
\end{aligned}$$

and similarly for $ef = \tau$.

We leave the $[M,[M,M]]$ cases for the masochists. The following seems like a good sequence of reductions:

1. Determine the dimension of the span of the three roots involved. If it's 1 or 2, there aren't so many cases to check.
2. If it's 3, we can assume all three roots are positive. Break into cases according to whether the system is A_1^3, $A_1 \times A_2$, or A_3.

One knows in advance in which root space any (intermediate) term will lie, which allows one to prune many terms that are obviously zero.

Here's one example. Let $e, f \in \hat{\Delta}$ map under π to the two simple roots of A_2, so that $g := (ef)^{-1}$ maps to the lowest root. Then

$$\begin{aligned}[\vec{e},[\vec{f},\vec{g}]] + [\vec{f},[\vec{g},\vec{e}]] + [\vec{g},[\vec{e},\vec{f}]] &= [\vec{e},\vec{fg}] + [\vec{f},\vec{ge}] + [\vec{g},\vec{ef}] \\ &= \pi(e) + \pi(f) + \pi(g) = 0.\end{aligned}$$

Richard assures me that one can avoid the case-checking by defining the "vertex operator algebra" containing the Lie algebra as the degree 1 piece (in particular, the VOA uses all of $\hat{L}$, not just $\hat{\Delta}$). This sounded like more trouble than it was worth for our purposes.

2. Twisting

Exercise. For each of the Dynkin diagrams, determine the automorphism group.

Note that only the simply-laced ones have any automorphisms![1]

Exercise. Show that the automorphism group of a root system is the semidirect product of the Weyl group and the diagram automorphism group.

Exercise. (Kostant's silly theorem) If the Dynkin diagram of a Lie group has no automorphisms, show that every representation of the Lie group is isomorphic to its dual. (First determine the highest weight of V_λ^*.)

Exercise. (Hard) Show the converse fails, making this a silly theorem.

Given a Dynkin diagram and an automorphism thereof, we can use the previous construction to lift it to an automorphism of the corresponding Lie algebra.

Given an order k automorphism Υ of a simply-laced diagram D, we get an automorphism of the "simple system", the set of simple roots. Define the **twisted simple system** as follows: for each Υ-orbit $\{\Upsilon^i \cdot \alpha\}$, make a new "simple root" $\sum_{i=1}^k \Upsilon^i \cdot \alpha$. Careful: this is *not* just the sum of the elements in the orbit, but rather that times $k/|\mathrm{stab}(\alpha)|$. (We will see below why this does really give a system of simple roots, by identifying the Lie algebra of which it is the simple system.)

From the simple system one can construct the root system, either by closing up under reflection or formally playing the find-the-highest-root game on the associated "twisted" Dynkin diagram.

[1] In fact the $B_2 = C_2$ and F_4 diagrams *do* have an automorphism when one is working in characteristic 2, and G_2 does in characteristic 3. This subtlety becomes important when one wants to classify the finite simple groups, all but 26 of whom are essentially Lie groups over finite fields.

Exercise. If Υ is nontrivial, show that its fixed points give the long simple roots in the twisted Dynkin diagram.

Exercise. If Υ is the nontrivial automorphism of the diagram A_n, show that the twisted diagram is $C_{(n+1)/2}$ if n is odd, $B_{n/2}$ if n is even.

Exercise. What does one get in all the other cases? In particular, note that every non-simply-laced Dynkin diagram arises this way!

Theorem. *Let Υ be an automorphism of a simply-laced Dynkin diagram D, and also use Υ to denote the induced automorphism on the Lie algebra $\mathfrak{g}$ constructed from D. Then the twisted diagram D^Υ is the Dynkin diagram for the fixed-point subalgebra $\mathfrak{g}^\Upsilon$.*

Unfortunately, we haven't built up the purely Lie algebra side of root systems enough to prove this (or even really define "the Dynkin diagram of a Lie algebra").