CHARACTER THEORY OF FINITE GROUPS
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These proofs are very sketchy! This is only meant to serve as reminder.

Fix G a finite group. Define a (linear) **representation** of G as a homomorphism of G to $GL(V)$, where V is a finite-dimensional complex vector space, and $GL(V)$ the group of invertible endomorphisms of V. Denote the homomorphism "$|_V$", i.e. each group element $g \in G$ is associated a linear transformation $g|_V : V \to V$. We will leave off the $|_V$ whenever it is possible to do so without confusion.

Theorem. *Let V be a rep of G. Then there exists a Hermitian inner product $\langle , \rangle$ on V preserved by G, i.e. for all $\vec{v}, \vec{w}$*

$$\langle \vec{v}, \vec{w} \rangle = \langle g\vec{v}, g\vec{w} \rangle.$$

Proof. Take a random Hermitian inner product and average it over G. □

In particular, if we pick an orthonormal basis w.r.t. one of these and write down the matrices $g|_V$ in that basis, they are unitary, hence diagonalizable. In fact the eigenvalues are necessarily roots of unity.

A **subrepresentation** or **invariant subspace** $W \leq V$ is one such that $G \cdot W \leq W$. If V has no subreps other than 0 and V, it is **irreducible**.

Given two reps W, U, we can make the direct sum $W \oplus U$ into a rep in an obvious way, with $W \oplus 0$ and $0 \oplus U$ as subreps. Being a direct sum, called **decomposable**, therefore implies reducibility, but for infinite groups the reverse isn't true. Luckily, we're not dealing with infinite groups.

Theorem. *Every rep of G is the direct sum of irreducibles.*

Proof. Given a subrep, we need to cook up an invariant complement. Fix an invariant Hermitian inner product, and take the perp to the subrep, which gives a complement that turns out to also be a subrep. Continue decomposing until done (by finite dimensionality). □

Given a rep V, define $\pi_V : V \to V$ as the sum $\frac{1}{|G|}\sum_g g|_V$. This has several easy properties:

1. $\forall g \in G, g|_V \pi_V = \pi_V$
2. $\pi_V^2 = \pi_V$
3. The image of π_V is $V^G := \{\vec{v} : g\vec{v} = \vec{v} \forall g \in G\}$, the subspace of **invariant vectors**.

Therefore

$$\dim V^G = \operatorname{Tr} \pi = \frac{1}{|G|} \sum_g \operatorname{Tr}(g|_V).$$

Let $\mathrm{Hom}(V, W)$ be the space of linear maps from V to W. If V, W are reps of G, define an action of G on $\mathrm{Hom}(V, W)$ by

$$g|_{\mathrm{Hom}(V,W)}\tau := g|_W \circ \tau \circ g|_V^{-1}$$

Then $\mathrm{Hom}(V, W)^G =: \mathrm{Hom}_G(V, W)$, the space of **equivariant maps** from V to W. An invertible equivariant map is called an **isomorphism**, and if one exists the two reps are called **isomorphic**.

Lemma. $\mathrm{Tr}\, g|_{\mathrm{Hom}(V,W)} = \overline{\mathrm{Tr}\, g|_V}\mathrm{Tr}\, g|_W$.

Proof. Pick eigenbases for g acting on V and W. The maps taking a basis vector of V with eigenvalue say λ to one of W with eigenvalue μ, and killing all others, are then a g-eigenbasis of $\mathrm{Hom}(V, W)$, with eigenvalue $\lambda^{-1}\mu$. Since g's eigenvalues are norm one complex numbers that's $\overline{\lambda}\mu$. Adding those up, the result follows. □

Lemma (Schur's lemma)**.** *Let V, W be irreducible representations ("irreps") of G. Then $\dim \mathrm{Hom}_G(V, W)$ is 1 if $V \cong W$, 0 if not.*

Proof. The kernel and image of these maps are subreps, therefore 0 or the whole thing by irreducibility. If there are two maps $V \to W$ and one is invertible, then by composing one with the inverse of the other we get an equivariant map $V \to V$. Subtract off the identity map times an eigenvalue, and this new equivariant map has a nonzero kernel, therefore it's the zero map. □

Theorem. *Let V be an irrep of G, $W = W_1 \oplus \ldots \oplus W_n$ a representation built as a sum of irreps W_i. Then the number of W_i isomorphic to V is*

$$\frac{1}{|G|}\sum_{g\in G} \overline{\mathrm{Tr}\,(g|_V)}\,\mathrm{Tr}\,(g|_W).$$

Proof. We prove something stronger:

$$\dim \mathrm{Hom}_G(V, W) = \dim \mathrm{Hom}(V, W)^G$$

$$= \dim \mathrm{Hom}_G(V, W)$$

$$= \frac{1}{|G|}\sum_{g\in G} \overline{\mathrm{Tr}\,(g|_V)}\,\mathrm{Tr}\,(g|_W).$$

We can recover a map from V to W from the individual projections to W_i. Most of these are automatically zero, but some of them give one degree of freedom (those $W_i \cong V$), by Schur's lemma. □

To state the corollaries, we need the **character** $g \mapsto \mathrm{Tr}\,(g|_V)$ of a representation V.

Corollary.
- *(Jordan-Hölder) Two sums of irreps are isomorphic as reps iff the sums have the same irreps occurring with the same multiplicities, just possibly in different order.*
- *Two reps are isomorphic iff they have the same character.*
- *Characters of irreps are orthonormal w.r.t. the obvious Hermitian inner product for functions on G (rescaled by $1/|G|$).*

From what we have, we can already show

Proposition. *The number of irreps of* G *is at most the number of conjugacy classes.*

Proof. The character of an irrep is a **class function** on G (which means one constant on conjugacy classes), since $\operatorname{Tr}(XYX^{-1}) = \operatorname{Tr} Y$. Since the characters of irreps are orthonormal (w.r.t. the usual inner product on $\operatorname{Fun}(G)$, rescaled by $1/|G|$), they are linearly dependent. Therefore the number of them is $\leq$ the dimension of the space of class functions, which is obviously the number of conjugacy classes. $\square$

We now give two ways to find out that these are equal. Both proceed via the regular representation.

1.1. **Analyzing the group algebra.** Given a decomposition $W = V_1 \oplus \ldots \oplus V_n$ of a representation into irreducibles, and a fixed irreducible V, define the **V-isotypic component of** W as the sum of those V_i that are isomorphic to V. Plainly W is the direct sum of its V-isotypic components, as V ranges over the set of isomorphism classes of irreducibles.

We show now that the isotypic components are canonical (do not depend on choice of decomposition), and how to pick them out:

Theorem 1. *Let* $\pi_{W,V} = \frac{\dim V}{|G|} \sum_g \overline{\operatorname{Tr}(g|_V)} g|_W$. *Then* $\pi_{W,V}$ *is a projection, and its image is the* V*-isotypic component of* W.

Proof. First observe that $\pi_{W,V}$ preserves each G-invariant subspace, and commutes with each $h|_W$:

$$h|_W \frac{\dim V}{|G|} \sum_g \overline{\operatorname{Tr}(g|_V)} g|_W h|_W^{-1} = \frac{\dim V}{|G|} \sum_g \overline{\operatorname{Tr}(g|_V)} (hgh^{-1})|_W$$
$$= \frac{\dim V}{|G|} \sum_g \overline{\operatorname{Tr}(hgh^{-1}|_V)} (hgh^{-1})|_W = \pi_{W,V}$$

So it preserves each V_i, and acts as a scalar there by Schur's lemma.

To determine by what scalar $\pi_{W,V}$ acts on V_i, take its trace:

$$\operatorname{Tr} \pi_{W,V}|_{V_i} = \frac{\dim V}{|G|} \sum_g \overline{\operatorname{Tr}(g|_V)} \operatorname{Tr} g|_{V_i} = (\dim V) \dim \operatorname{Hom}_G(V, V_i)$$

So either 0 if they aren't isomorphic, or $\dim V_i$ if they are, which is to say $\pi_{W,V}$ is the identity on those irreducible components isomorphic to V. $\square$

You should think of isotypic components as like decomposing into eigenspaces for a diagonalizable matrix, vs. picking an eigenbasis; only the first is canonical.

Define the **group algebra** $\mathbb{C}[G]$ as the set of formal linear combinations $\sum_g \lambda_g g, \lambda_g \in \mathbb{C}$ of group elements. Inside this we locate G as a set, $g \leftrightarrow 0 + 0 + 0 + 1g + 0 + \ldots + 0$. Define the multiplication $\cdot$ on $\mathbb{C}[G]$ in the unique way that extends the group law $g \cdot h = gh$. Note that this vector space is naturally a representation of $G \times G$ by left and right multiplication:

$$(g, h)|_{\mathbb{C}[G]} \sum_f \lambda_f f := \sum_f \lambda_f gfh^{-1}.$$

Since G is a subgroup of $\mathbb{C}[G]^\times$, every module over $\mathbb{C}[G]$ is naturally a G-representation, and by linear extension the reverse is true. It is equally easy to see that 2-sided ideals in $\mathbb{C}[G]$ are the same thing as $G \times G$-invariant subspaces.

To each irreducible V, let I_V denote the 2-sided ideal of $\mathbb{C}[G]$ arising as the kernel of the ring homomorphism $\mathbb{C}[G] \to \mathrm{End}(V)$.

Theorem. *Let $W = \sum_i W_i$ be a representation of G, where the $\{W_i\}$ are irreducible.*

1. *The image of $\mathbb{C}[G] \to \mathrm{End}(W)$ lands inside the subring $\oplus \mathrm{End}(W_i)$.*
2. *The kernel is the intersection of those those I_V for V isomorphic to some W_i.*
3. *The intersection of all the I_V is zero.*
4. *The number of irreducible representations in G is equal to the number of conjugacy classes of G.*

Proof. 1. This is just the statement that G-subreps are $\mathbb{C}[G]$-submodules.
2. An element is in the kernel iff it is in the kernel of each subsequent projection $\mathbb{C}[G] \to \mathrm{End}(W_i)$.
3. Let W be the left regular representation of $\mathbb{C}[G]$, whose kernel is trivial.
4. Let $W = \bigoplus V_i$ be the direct sum of one copy each of the irreps of G. The map $\mathbb{C}[G] \to \bigoplus \mathrm{End}(V_i)$ is, by the above, injective. It is also a G-equivariant map, where G acts on $\mathbb{C}[G]$ by conjugation, and on each $\mathrm{End}(V_i)$ by the definition given before for $\mathrm{Hom}(V_i, V_i)$. Therefore $\mathbb{C}[G]^G$ injects into $\bigoplus \mathrm{End}(V_i)^G$.

   What is $\mathbb{C}[G]^G$? It is the center of $\mathbb{C}[G]$, and is easily seen to be the elements $\sum_g \lambda_g g$ where $\lambda_g : G \to \mathbb{C}$ is a class function. Therefore $\dim \mathbb{C}[G]^G$ is the number of conjugacy classes. Whereas by Schur's lemma, each $\mathrm{End}(V_i)^G$ is just the 1-dimensional space of scalars, so $\dim \bigoplus \mathrm{End}(V_i)^G$ is the number of irreps. The result follows.

□

Define the **character table** as the matrix whose rows correspond to isomorphism classes of irreps V, whose columns to the conjugacy classes $[g]$, and whose $(V, [g])$th entry is $\mathrm{Tr}\,(g|_V)$.

Corollary. *The character table is a square matrix. If you multiply the $[g]$th column by $\sqrt{|[g]|/|G|}$, it becomes a unitary matrix. In particular the columns are already orthogonal, and the norm square of the $[g]$ column is $|G|/|[g]|$.*

Given a G-rep and an H-rep, the space $\mathrm{Hom}(V, W)$ is naturally a $G \times H$-module: $(g, h) \cdot \tau := g \circ \tau \circ h^{-1}$. This is called the **outer Hom-space**. (Our previous definition, which needlessly asked that $G = H$, really took this definition and looked at the action of the diagonal G inside $G \times G$.)

Corollary (special case of either Peter-Weyl or Artin-Wedderburn)**.** *The space $\mathbb{C}[G]$ is isomorphic to the direct sum $\bigoplus \mathrm{Hom}(V_i, V_i)$ of full matrix algebras (summing over one copy each of G's irreps), as an algebra and also as a $G \times G$-representation.*

Proof. Since $\bigoplus V_i$ is a rep of G, we have a map $\mathbb{C}[G] \to \bigoplus \mathrm{Hom}(V_i, V_i)$ which is an algebra map and one checks the definitions to see that it is tautologically a $G \times G$-equivariant map. We proved before that it's injective. Now look at the norm square of the first column of the character table (by tradition, reserved for the singleton conjugacy class of the identity) to see that the dimensions are equal. □

Peter-Weyl says that the same is true for all compact groups G, where $\mathbb{C}[G]$ is replaced by $L^2(G)$ and direct sum by L^2 direct sum. Artin-Wedderburn says that the same is true for all semisimple Artinian algebras over an algebraically closed field. In particular one can use the more general case of Artin-Wedderburn to understand the theory of *real* vector space representations of G.

1.2. **Analysing the regular representation.** Forgetting this analysis of ideals, we can just think about $\mathbb{C}[G]$ as a representation of G (acting on the left).

Then each irrep V appears in $\mathbb{C}[G]$:

$$\dim \operatorname{Hom}_G(V, \mathbb{C}[G]) = \frac{1}{|G|}\sum_g \overline{\operatorname{Tr} g|_V}\operatorname{Tr} g|_{\mathbb{C}[G]} = \frac{1}{|G|}\overline{\operatorname{Tr} e|_V}\,|G| = \dim V.$$

One property of outer Homs we didn't mention before it that they're irreducible (if the contituents are).

$$\frac{1}{|G\times H|}\sum_{(g,h)}|\operatorname{Tr}(g,h)|_{\operatorname{Hom}(V,W)}|^2 = \frac{1}{|G||H|}\sum_{(g,h)}|\operatorname{Tr} g|_V|^2|\operatorname{Tr} h|_W|^2$$
$$= (\frac{1}{|G|}\sum_g |\operatorname{Tr} g|_V|^2)(\frac{1}{|H|}\sum_h |\operatorname{Tr} h|_W|^2) = 1\cdot 1 = 1.$$

We don't yet know that these give *all* irreducibles of $G \times H$, though this will indeed turn out to be the case.

Nonetheless, we can figure out which of these occur in $\mathbb{C}[G]$ considered as a $G\times G$-rep:

$$\frac{1}{|G|^2}\sum_{(g,h)}\overline{\operatorname{Tr}(g,h)|_{\operatorname{Hom}(V,W)}}\operatorname{Tr}(g,h)|_{\mathbb{C}[G]} = \frac{1}{|G|^2}\sum_{(g,h)}\operatorname{Tr}(g|_V)\overline{\operatorname{Tr} h|_W}\sum_a [1 \text{ if } gah^{-1} = a, 0 \text{ if not}]$$
$$= \frac{1}{|G|^2}\sum_g \operatorname{Tr}(g|_V)\sum_a\sum_h [1 \text{ if } a^{-1}ga = h, 0 \text{ if not}]\overline{\operatorname{Tr} h|_W}$$
$$= \frac{1}{|G|^2}\sum_g \operatorname{Tr}(g|_V)\sum_a \overline{\operatorname{Tr}(a^{-1}ga)|_W} = \frac{1}{|G|^2}\sum_g \operatorname{Tr}(g|_V)\sum_a\overline{\operatorname{Tr} g|_W} = \frac{1}{|G|}\sum_g \operatorname{Tr}(g|_V)\overline{\operatorname{Tr} g|_W}$$

is 1 if $V\cong W$, 0 if not.

In particular $\mathbb{C}[G]$ contains $\oplus_V \operatorname{Hom}(V,V)$ as a $G \times G$-subrep. But by comparing dimensions (using the above calculation of $\mathbb{C}[G]$ as just a left G-rep), we know that they must in fact be equal.

From there we can show that the number of conjugacy classes equals the number of irreps by the same argument as before.

Corollary. *Every irrep of* $G\times H$ *is of the form* $\operatorname{Hom}(V,W)$ *for* V,W *irreps of* G,H *respectively.*

Proof. The number of conjugacy classes is the product of the individual numbers. Each of these we showed irreducible (and the same sort of character argument shows they are orthogonal, hence distinct). So this already produces the right number of irreducibles. □