NOTES #2: HILBERT AND KRULL DIMENSIONS

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1. Associated gradeds and their geometry

A **decreasing ($\mathbb{N}$-)filtration** of a ring S is a sequence $S = S_0 \geq S_1 \geq S_2 \geq \ldots$ of abelian subgroups, such that $S_i S_j \leq S_{i+j}$. In particular each S_m is automatically an ideal and contains S_1^m. The quickest-decreasing such filtration, with S_1 a given ideal I, has $S_m = I^m$ and is called the **I-adic filtration**.

Then **associated graded** $\mathrm{gr}\, S := \oplus_n S_m/S_{m+1}$ inherits a natural (graded) ring structure, with $(\mathrm{gr}\, S)_m$ the mth summand.

Sometimes S will already be a graded ring, say with a Hilbert function h_S, and I a graded ideal of S. In that case $\mathrm{gr}\, S$ is bigraded, and $h_{\mathrm{gr}\, S} = h_S$.

If $J \leq S$ is an ideal, then we can consider $\mathrm{gr}\, J \leq \mathrm{gr}\, S$, automatically a graded ideal. For example, if $J = \langle p \rangle$ is a principal ideal in a polynomial ring S which is m-adically filtered (where m is the "irrelevant" ideal, generated by the variables), then $\mathrm{gr}\, S \cong S$, but under this isomorphism $\mathrm{gr}\, J$ becomes the principal ideal generated by the lowest homogeneous part of p.

1.1. Geometry; not ready yet.

Let $J \leq \mathbb{C}[x_1, \ldots, x_n, s] =: R$ which we filter by powers of $\langle s \rangle$. How can we think about $\mathrm{gr}\, J$, in ways that let us convert to geometry? We want a construction that picks the lowest powers of s out of every polynomial in J.

First, introduce an extra variable t and consider the ring map

$$\begin{array}{ccc} R & \stackrel{\alpha^*}{\to} & R[t] \\ x_i & \mapsto & x_i \\ s & \mapsto & st \end{array}$$

which, on spaces, amounts to a group action rescaling the last coordinate:

$$\begin{array}{ccc} \mathbb{C}^n & \stackrel{\alpha}{\longleftarrow} & \mathbb{C}^n \times \mathbb{C} \\ (x_1, \ldots, x_n, st) & \leftarrow\!\shortmid & (x_1, \ldots, x_n, s, t) \end{array}$$

Now take the ideal J_+ generated inside $R[t]$ by $\alpha^*(J)$, giving us the pushout diagram

$$\begin{array}{ccc} R & \stackrel{\alpha^*}{\to} & R[t] \\ \downarrow & & \downarrow \\ R/J & \stackrel{\alpha^*}{\to} & R[t]/J_+ \end{array}$$

Date: September 5, 2017.

This J_+ is the smallest possible ideal with which to complete this square. Correspondingly, the pullback diagram on spaces has the largest possible SE corner:

$$\begin{array}{ccccl} \mathbb{C}^n & \stackrel{\alpha}{\longleftarrow} & \mathbb{C}^n \times \mathbb{C} & & \\ \uparrow & & \uparrow & & \\ V(J) & \longleftarrow & V(J_+) & = & \{(x,z) \; : \; zx \in V(J), z \in \mathbb{C}^\times\} \end{array}$$

In particular, $V(J_+) \cap \{z = t\}$ for $t \neq 0$ gives $t^{-1} \cdot V(J)$, stretching out the last coordinate by t^{-1}.

Now we want to take $t = 0$, penalizing all the monomials of non-minimal s-degree. But that doesn't quite work...

2. Hilbert dimension

Theorem 2.1. *Let* $J \leq R$ *be a graded ideal, and* $H\dim(R/J)$ *its Hilbert dimension. Then* $H\dim(R/J) = H\dim(R/\sqrt{J})$*, i.e.* $H\dim$ *is a set-theoretic notion.*

Proof. If $J = \sqrt{J}$ we're done. Else there is a nonzero nilpotent r. Its lowest graded component is also nilpotent of the same (or lesser) order, so we can assume r is homogeneous. Let s be its last nonzero power, so $s \neq 0$ but $s^2 = 0$. Let $S := R/J$. Then from

$$0 \to \langle s \rangle \to S \to S/\langle s \rangle \to 0 \qquad 0 \to ann_S(s) \to S \to S[\deg s] \to 0 \qquad \langle s \rangle \leq ann_S(s)$$

we derive

$$h_S = h_{S/\langle s \rangle} + \langle s \rangle \leq h_{S/\langle s \rangle} + ann_S(s) = h_{S/\langle s \rangle} + h_S - h_{S[\deg s]}$$
$$h_{S[\deg s]} \leq h_{S/\langle s \rangle} \leq h_S$$

where $\leq$ means "at each value". The $[\deg s]$ shift doesn't change the degree of the polynomial, so $H\dim(S) = H\dim(S/\langle s \rangle)$.

As we mod out nilpotents, we get larger ideals contained inside $\sqrt{J}$, all with the same $H\dim$. Such an ascending chain must terminate, at $\sqrt{J}$. □

Since it's a set-theoretic notion it's reasonable to define the Hilbert dimension of a Zariski-closed subset of $\mathbb{CP}^n$.

Theorem 2.2. *Let* $ab = 0$ *be zero divisors in* $S = R/J$*, where* J, a, b *are homogeneous. Then* $H\dim(R/J) = \max\{H\dim(R/a), H\dim(R/b)\}$.

Proof. First safely replace J with $\sqrt{J}$ (which might kill a or b, but anyway they'll still be zero divisors).

Then consider the map $S \to S/a \oplus S/b$. The kernel consists of s of the form $ma = nb$. But then $s^2 = manb = mnab = 0$, so $s = 0$. Consequently

$$h_S \leq h_{S/a} + h_{S/b} \qquad \leq 2h_S$$

and these inequalities on nonnegative-valued polynomials establish the claim. □

In particular, if $J = \bigcap P$ the minimal primes containing P (which is equivalent to J radical), then $H\dim(R/J) = \max_P H\dim(R/P)$. Set-theoretically, this says $H\dim\ V(J) = \max_P H\dim\ V(P)$, which feels very reasonable.

For the next theorem, we need to define the $H\dim$ to be -1 when the Hilbert polynomial is actually 0.

Theorem 2.3. *Let $s \in S$ be a homogeneous non-zero-divisor of degree $k > 0$. Then* $\mathrm{H}\dim(S/s) = \mathrm{H}\dim(S) - 1$.

Proof. The SES $0 \to S[-k] \to S \to S/s \to 0$ gives $h_{S/s} = h_S - h_{S[-k]}$, and this difference is of one lower polynomial degree. □

Corollary 2.4. *Let R/J be a domain, and X an algebraic set properly contained in $V(J)$. Then* $\mathrm{H}\dim\ X < \mathrm{H}\dim\ V(J)$.

Proof. Since it's an algebraic set, it's $V(K)$ for some $K \geq J$. By the proper containment, $\exists k \in K \setminus J$, i.e. $\bar{k} \neq 0 \in R/J$. Since R/J is a domain, $\bar{k}$ is a nonzero divisor. Therefore $\mathrm{H}\dim(R/K) \leq \mathrm{H}\dim(R/K + \langle k \rangle) < \mathrm{H}\dim(R/J)$. □

These last few results focus attention on domains in order to understand dimension. Define the **Krull dimension** $\dim S$ of S as the maximum k such that there exists a sequence of graded surjections

$$S \twoheadrightarrow D_k \twoheadrightarrow D_{k-1} \twoheadrightarrow \ldots \twoheadrightarrow D_0 \twoheadrightarrow 0$$

where the D_i are domains, and no map $D_i \twoheadrightarrow$ is an isomorphism. This behaves much like Hilbert dimension:

Theorem 2.5. $\dim R/J = \dim R/\sqrt{J} = \max_P \dim R/P$ *where P varies over minimal prime ideals containing J.*

Proof. The first map $R/J \twoheadrightarrow D_k$ must factor through $R/\sqrt{J}$, so they have the same Krull dimension. Then the kernel gives a prime ideal $P \geq J$, and it's obviously better to have a smaller P than larger when trying to maximize k. □

Incidentally, a **subvariety** $V(J)$ **of affine (resp. projective) space** is defined as the vanishing set of a prime ideal (resp. projective vanishing set of a homogeneous prime ideal).

Let $e : k[x_0, \ldots, x_n] \twoheadrightarrow k[x_1, \ldots, x_n]$ take $x_0 \mapsto 1$, and given $J \leq k[x_1, \ldots, x_n]$ an ideal, define the **homogenization** $\tilde{J}$ of J as the maximal homogeneous ideal contained in $e^{-1}(J)$, i.e. $\oplus_n (e^{-1}(J) \cap R_n)$.

Lemma 2.6. $e(\tilde{J}) = J$, *and if J is prime then so is $\tilde{J}$.*

Proof. Tautologically $e(\tilde{J}) \leq J$. Given $p \in J$, we can homogenize it w.r.t. x_0 to get an element $\tilde{p} \in \tilde{J}$, and then $e(\tilde{p}) = p$. Relatedly, we claim $\tilde{J} : \langle x_0 \rangle = \tilde{J}$.

Now assume $ab \in \tilde{J}$, but $a, b \notin \tilde{J}$, for contradiction. Write $a = a_{min} + a_{rest}$, if a is not homogeneous, b similarly. Then $a_{min} b_{min} = (ab)_{min} \in \tilde{J}$. If $a_{min} \in \tilde{J}$, then we can replace a by $a - a_{min}$ and start over. Repeat until $a_{min}, b_{min} \notin \tilde{J}$, then replace a, b by a_{min}, b_{min} in order to reduce to the case that they're homogeneous. Moreover, we can assume $x_0 \not\div a, b$ using $\tilde{J} : \langle x_0 \rangle = \tilde{J}$.

Since $e(ab) \in J$, we know $e(a), e(b) \in J$, so their homogenizations $\widetilde{e(a)}, \widetilde{e(b)} \in \tilde{J}$. But two homogeneous polynomials with the same dehomogenization $x_0 \mapsto 1$, neither one a multiple of x_0, must be equal. □

Theorem 2.7. *For $J \leq R$ a graded ideal,* $\mathrm{H}\dim(R/J) = \dim(R/J) - 1$.

Proof. We define a third notion: the **graded Krull dimension** $g\dim(R/J)$ is the length of a maximal chain, using *homogeneous* prime ideals $\geq J$. So tautologically $g\dim(R/J) \leq \dim(R/J)$. We want to show $g\dim(R/J) \geq H\dim(R/J) \geq \dim(R/J)$.

For the first inequality, we need to construct a long chain of homogeneous prime ideals. Since $\dim S = \max_{P\geq J} \dim R/P$, we start by picking a P achieving that maximum with which to define D_k. So $\dim S = \dim R/P$ and $H\dim S = H\dim R/P$, i.e. we reduce to the case of domains.

If $D_k = \mathbb{C}$ we're done, else pick $r \neq 0$ graded of positive degree, and $Q \geq \langle r\rangle$ a prime ideal s.t. $H\dim D_k/Q = H\dim D_k/r = H\dim D_k - 1$. This is a graded domain of (one) smaller Hilbert dimension, and by induction it has a chain of graded domain quotients of the right length.

For the second inequality, let $S \twoheadrightarrow D_k \twoheadrightarrow D_{k-1} \twoheadrightarrow \ldots \twoheadrightarrow D_0 \twoheadrightarrow 0$ be a maximal chain, and $J \leq P_k < \ldots < P_0 < R$ the corresponding chain of prime ideals. If we z-homogenize these, we get a chain $\tilde{J} \leq \tilde{P}_k < \ldots < \tilde{P}_0 < R[z]$, and by the lemma these $\tilde{P}_i$ are again prime (but now graded). □

The Hilbert dimension is easier to work with in that it doesn't require any choices to be made (then maximized) for its definition. Also, we can compute it using Gröbner basis techniques (associated gradeds). Conversely, the Krull dimension can be defined for arbitrary commutative rings.

3. Bézout's theorem

Theorem 3.1. *Let* p, q *be two homogeneous polynomials in* x, y, z *with* $\gcd(p, q) = 1$*, and* $S = k[x, y, z]/\langle p, q\rangle$*. Then* $H\dim(S) = 0$, $\deg(S) = \deg(p)\deg(q)$.

Proof. By the condition on $\gcd$, q is not a zero divisor in $k[x, y, z]/\langle p\rangle$, and of course p isn't one in $k[x, y, z]$. So modding them out decreases the dimension by 1 each time, and multiplies the degree by the degree. □

Exercise.

(1) Let $p_1, \ldots, p_d$ be homogeneous polynomials in $R = k[x_0, \ldots, x_n]$.
Show that $H\dim R/\langle p_1, \ldots, p_d\rangle \geq n - d$, and if equality holds, then
$\deg(R/\langle p_1, \ldots, p_d\rangle) = \prod_i \deg(p_i)$. Such ideals are called **complete intersections**.

(2) Let $M \subseteq \mathbb{R}^n$ be a fiber over a regular value of a map $p : M \to \mathbb{R}^d$. Show that the normal bundle to M inside $\mathbb{R}^n$ is trivial, e.g. a Möbius strip does not embed in $\mathbb{R}^n$ as a closed complete intersection.