Given a cone $C \leq \mathbb{R}^n$ defined by finitely many integral linear inequalities, we define $TV_C := \text{Specm} \mathbb{C}[C \cap \mathbb{Z}^n]$ as its associated toric variety.

#1. Let $F \leq C$ be a face of $C$, i.e. where some of those inequalities are actually equalities. Show that $TV_F \hookrightarrow TV_C$ by identifying $F$’s ring with the quotient of $C$’s by an ideal.

#2. Let $S$ be the blowup algebra, blowing up $\mathbb{C}^2$ at the origin. Find a cone $C$ whose associated algebra is $S$. (It’ll be 3-dimensional, because $TV_C$ will be the affine cone over $\widetilde{\mathbb{C}^2}$, not $\widetilde{\mathbb{C}^2}$ itself.)

#3. Let $S$ be the blowup algebra blowing up $\mathbb{C}^3$ along $xy = 0$. Find a cone $C$ whose associated algebra is $S$.

#4. Determine the singular locus of $\text{Specm} S$ and $\text{Projm} S$ (from the question above).

#5. Let $S, T$ be two finitely generated algebras over $\mathbb{C}$ (not necessarily coming from cones). Relate the spaces $\text{Specm} S, \text{Specm} T, \text{Specm} (S \otimes_{\mathbb{C}} T)$, and also relate their singular loci.