1. Let $G \geq H$ be a group and subgroup. Define
$$K := \{g \in G : gH = Hg\}$$
a [10]. Prove that $K$ is a subgroup.
Answer. $eH = H = He$, so $e \in K$.
If $g \in K$ so $gH = Hg$, we multiply on both sides by $g^{-1}$, to find $Hg^{-1} = g^{-1}H$, so now $g^{-1} \in K$.
If $f, g \in K$, then $fgH = fHg = Hfg$, so $fg \in K$.

1b [15]. Prove that not only is $K \geq H$, but $H$ is a normal subgroup of $K$.
Answer. If $h \in H$, then $hH = H = Hh$, so $h \in K$. Hence $H \leq K$.
The condition “$kH = Hk \forall k \in K$” is one of the equivalent conditions for normality.
But if you prefer multiplying on the RHS by $k^{-1}$, you can instead say $kHk^{-1} = H \forall k \in K$.
If you’d rather go to elements individually, say $khk^{-1} \in H \forall k \in K, h \in H$.

1c [10]. Give an example where $H$ is not normal in $G$, and compute its $K$.
(An acceptable answer to “give an example” will go “Let $G$ be this specific group, and $H$ be this specific subgroup. Then $K$ is...”)
Answer. If $G$ were abelian, then all subgroups would be normal, so we need a nonabelian one. Our smallest example is $S_3$, and we’ve seen before that $H = \{e, (12)\} \leq S_3$ is not normal. (And now we’ll see it again.)
Look at $\pi \circ (12) \circ \pi^{-1} = (\pi(1) \pi(2))$. For $H$ to be normal, we need $\pi \circ (12) \circ \pi^{-1} \in H$, i.e. $(\pi(1) \pi(2)) = e$ or $(12)$. So either $\pi(1) = 1$ and $\pi(2) = 2$, or $\pi(1) = 2$ and $\pi(2) = 1$. In which case $\pi \in H$. It turns out that $K = H$ in this example.

2. Recall that a relation $R$ on $M$ is a set of ordered pairs, $R \subseteq M \times M$. So given two relations $R, S$ we can talk about $R \cap S, R \cup S$.
[10] Give an example of a set $M$ and two equivalence relations $R, S$ (write $\sim_R, \sim_S$ if you want) such that $R \cup S$ is not an equivalence relation.
Answer. In fact $R \cup S$ will be reflexive and symmetric, but if we take
$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\} \text{ i.e. generated by } 1 \sim_R 2$$
$$S = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\} \text{ i.e. generated by } 2 \sim_S 3$$
then $(1, 2), (2, 3) \in R \cup S$, but $(1, 3) \not\in R \cup S$, so it’s not transitive.

3. Let $\varphi : G \to H$ be a homomorphism of finite groups, and let $\text{ord}(g)$ denote the order of $g$, the smallest $n$ s.t. $g^n = e$.
a [10]. What’s the relation between $\text{ord}(g)$ and $\text{ord}(\varphi(g))$?
Answer. $\varphi(g)^{\text{ord}(g)} = \varphi(f^{\text{ord}(g)}) = \varphi(e_G) = e_H$, so $\text{ord}(\varphi(g))$ divides $\text{ord}(g)$.
3b [15]. Find all the homomorphisms \( \varphi : S_3 \rightarrow S_3 \) (and not just the automorphisms).

**Answer.** Where do \((12), (23)\) go? By 3a, they have to go to elements of order dividing 2. If \((12) \mapsto e\), i.e. \((12) \in \ker \varphi\), then \((23), (13) \mapsto e\) also, because they’re conjugate to \((12)\) and \(\ker \varphi\) is a normal subgroup. (Then \((123), (132)\) must also since they’re in the subgroup generated by \((12), (23)\).) So we’ve found one of the homomorphisms: everybody goes to \(e\). Now assume we’re not in that case...

So \((12)\) goes to some element of order 2, i.e. \((ab) \in \{(12), (13), (23)\}\). If \((23)\) goes to \(e\), we’ve got a contradiction by the argument in the previous paragraph. If \((23)\) goes to \((ab)\) too, then \((123), (132) \mapsto e\) and \((13) \mapsto (ab)\). So we get three more homomorphisms: you can think of them as \(S_3 \xrightarrow{\text{sign}} \{\pm 1\} \xrightarrow{} S_3\), with \(-1 \mapsto (ab)\).

Final case: \((12) \mapsto (ab)\), and \((23)\) goes to somebody else \((ac) = (ca)\). (Two subsets of \{1, 2, 3\} of size two must overlap; for convenience we switch \(a \leftrightarrow b\), \(a \leftrightarrow c\) as needed, to make the overlap be the first one in each pair.) Now the map is \(\pi \mapsto (bac)\pi(bac)^{-1}\). There are 3! such maps.

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4. Let \(G\) be a product of cyclic groups, \(\prod_{i=1}^{f} \mathbb{Z}_{n_i}\), for some \(n_1, \ldots, n_f \in \mathbb{N}_+\).

Recall the Chinese Remainder Theorem: if \(\gcd(m, n) = 1\), then \(\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n\).

a [15]. Prove that there is another sequence \(a_1, \ldots, a_g\) with \(G \cong \prod_{i=1}^{g} \mathbb{Z}_{a_i}\), but now in addition, we have \(a_i|a_{i+1}\) for each \(i = 1, \ldots, g - 1\).

4b [15]. Show that \(g \leq f\).

**Answer.** Use \(\text{CR3}\) to break the \(f\) cyclic groups into products of cyclic groups each of prime power order. Call this the Big Breakup. So each prime \(p\) occurs in at most \(f\) factors in the Big Breakup, after we’ve finished breaking.

There will be some prime \(p\) that occurs the most times, \(g \leq f\). Then let \(a_{g+1-i} = \prod_p (i\text{th smallest power of } p\text{ occurring in the Big Breakup}, \text{or } 1 \text{ if } p \text{ occurred fewer than } i \text{ times}).\)

Now use \(\text{CR3}\) to reassemble the Big Breakup back into the cyclic groups \(\mathbb{Z}_{a_i}\).

Here’s an example, to help see what’s going on: start with \(\mathbb{Z}_{54} \times \mathbb{Z}_{24} \times \mathbb{Z}_{14}\). Factor into prime powers:

\[
54 = 2^1 \times 3^3 \times 5^0 \times 7^0 \times \ldots
\]
\[
24 = 2^3 \times 3^1 \times 5^0 \times 7^0 \times \ldots
\]
\[
14 = 2^1 \times 3^0 \times 5^0 \times 7^1 \times \ldots
\]

Hence our group is \((\mathbb{Z}_2 \times \mathbb{Z}_{27}) \times (\mathbb{Z}_8 \times \mathbb{Z}_3) \times (\mathbb{Z}_2 \times \mathbb{Z}_7)\). Now sort the columns vertically:

\[
2^1 \times 3^0 \times 5^0 \times 7^0 \times \ldots
\]
\[
2^1 \times 3^1 \times 5^0 \times 7^0 \times \ldots
\]
\[
2^3 \times 3^3 \times 5^0 \times 7^1 \times \ldots
\]

Then reassemble:

\[
2 = 2^1 \times 3^0 \times 5^0 \times 7^0 \times \ldots
\]
\[
6 = 2^1 \times 3^1 \times 5^0 \times 7^0 \times \ldots
\]
\[
1512 = 2^3 \times 3^3 \times 5^0 \times 7^1 \times \ldots
\]

so \(\mathbb{Z}_{54} \times \mathbb{Z}_{24} \times \mathbb{Z}_{14} \cong \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{1512}\).