MATH 3340 MIDTERM, SPRING 2017

1. Let $G \geq H$ be a group and subgroup. Define

$$K := \{g \in G \ : \ gH = Hg\}$$

a [10]. Prove that K is a subgroup.

Answer. $eH = H = He$, so $e \in K$.
If $g \in K$ so $gH = Hg$, we multiply on both sides by g^{-1}, to find $Hg^{-1} = g^{-1}H$, so now $g^{-1} \in K$.
If $f, g \in K$, then $fgH = fHg = Hfg$, so $fg \in K$.

1b [15]. Prove that not only is $K \geq H$, but H is a normal subgroup of K.

Answer. If $h \in H$, then $hH = H = Hh$, so $h \in K$. Hence $H \leq K$.
The condition "$kH = Hk \; \forall k \in K$" is one of the equivalent conditions for normality.
But if you prefer multiplying on the RHS by k^{-1}, you can instead say $kHk^{-1} = H \; \forall k \in K$.
If you'd rather go to elements individually, say $khk^{-1} \in H \; \forall k \in K, h \in H$.

1c [10]. Give an example where H is not normal in G, and compute its K.

(An acceptable answer to "give an example" will go "Let G be this specific group, and H be this specific subgroup. Then K is...")

Answer. If G were abelian, then all subgroups would be normal, so we need a nonabelian one. Our smallest example is S_3, and we've seen before that $H = \{e, (12)\} \leq S_3$ is not normal. (And now we'll see it again.)

Look at $\pi \circ (12) \circ \pi^{-1} = (\pi(1)\ \pi(2))$. For H to be normal, we need $\pi \circ (12) \circ \pi^{-1} \in H$, i.e. $(\pi(1)\ \pi(2)) = e$ or (12). So either $\pi(1) = 1$ and $\pi(2) = 2$, or $\pi(1) = 2$ and $\pi(2) = 1$. In which case $\pi \in H$. It turns out that $K = H$ in this example.

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2. Recall that a relation R on M is a set of ordered pairs, $R \subseteq M \times M$. So given two relations R, S we can talk about $R \cap S$, $R \cup S$.

[10] Give an example of a set M and two equivalence relations R, S (write $\underset{R}{\sim}, \underset{S}{\sim}$ if you want) such that $R \cup S$ is not an equivalence relation.

Answer. In fact $R \cup S$ will be reflexive and symmetric, but if we take

$$R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\} \qquad \text{i.e. generated by } 1 \underset{R}{\sim} 2$$

$$S = \{(1,1), (2,2), (3,3), (2,3), (3,2)\} \qquad \text{i.e. generated by } 2 \underset{S}{\sim} 3$$

then $(1,2), (2,3) \in R \cup S$, but $(1,3) \notin R \cup S$, so it's not transitive.

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3. Let $\varphi : G \to H$ be a homomorphism of finite groups, and let $\mathrm{ord}(g)$ denote the **order** of g, the smallest n s.t. $g^n = e$.

a [10]. What's the relation between $\mathrm{ord}(g)$ and $\mathrm{ord}(\varphi(g))$?

Answer. $\varphi(g)^{\mathrm{ord}(g)} = \varphi(f^{\mathrm{ord}(g)}) = \varphi(e_G) = e_H$, so $\mathrm{ord}(\varphi(g))$ divides $\mathrm{ord}(g)$.

3b [15]. Find all the homomorphisms $\varphi : S_3 \to S_3$ (and not just the automorphisms).

Answer. Where do $(12), (23)$ go? By 3a, they have to go to elements of order dividing 2.

If $(12) \mapsto e$, i.e. $(12) \in \ker\varphi$, then $(23), (13) \mapsto e$ also, because they're conjugate to (12) and $\ker\varphi$ is a normal subgroup. (Then $(123), (132)$ must also since they're in the subgroup generated by $(12), (23)$.) So we've found one of the homomorphisms: everybody goes to e. Now assume we're not in that case...

So (12) goes to some element of order 2, i.e. $(ab) \in \{(12), (13), (23)\}$. If (23) goes to e, we've got a contradiction by the argument in the previous paragraph. If (23) goes to (ab) too, then $(123), (132) \mapsto e$ and $(13) \mapsto (ab)$. So we get three more homomorphisms: you can think of them as $S_3 \overset{\text{sign}}{\twoheadrightarrow} \{\pm 1\} \hookrightarrow S_3$, with $-1 \mapsto (ab)$.

Final case: $(12) \mapsto (ab)$, and (23) goes to somebody else $(ac) = (ca)$. (Two subsets of $\{1, 2, 3\}$ of size two must overlap; for convenience we switch $a \leftrightarrow b$, $a \leftrightarrow c$ as needed, to make the overlap be the first one in each pair.) Now the map is $\pi \mapsto (bac)\pi(bac)^{-1}$. There are 3! such maps.

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4. Let G be a product of cyclic groups, $\prod_{i=1}^{f} \mathbb{Z}_{n_i}$, for some $n_1, \ldots, n_f \in \mathbb{N}_+$.

Recall the Chinese Remainder Theorem: if $\gcd(m, n) = 1$, then $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$

a [15]. Prove that there is another sequence $a_1, \ldots, a_g$ with $G \cong \prod_{i=1}^{\not{f}\, g} \mathbb{Z}_{a_i}$, but now in addition, we have $a_i | a_{i+1}$ for each $i = 1, \ldots, g - 1$.

4b [15]. Show that $g \leq f$.

Answer. Use CRΘ to break the f cyclic groups into products of cyclic groups each of prime power order. Call this the Big Breakup. So each prime p occurs in at most f factors in the Big Breakup, after we've finished breaking.

There will be some prime p that occurs the most times, $g \leq f$. Then let $a_{g+1-i} = \prod_p$(ith smallest power of p occurring in the Big Breakup, or 1 if p occurred fewer than i times).

Now use CRΘ to reassemble the Big Breakup back into the cyclic groups $\mathbb{Z}_{a_i}$.

Here's an example, to help see what's going on: start with $\mathbb{Z}_{54} \times \mathbb{Z}_{24} \times \mathbb{Z}_{14}$. Factor into prime powers:

$$\begin{array}{rcccccccccl} 54 & = & 2^1 & \times & 3^3 & \times & 5^0 & \times & 7^0 & \times & \cdots \\ 24 & = & 2^3 & \times & 3^1 & \times & 5^0 & \times & 7^0 & \times & \cdots \\ 14 & = & 2^1 & \times & 3^0 & \times & 5^0 & \times & 7^1 & \times & \cdots \end{array}$$

Hence our group is $(\mathbb{Z}_2 \times \mathbb{Z}_{27}) \times (\mathbb{Z}_8 \times \mathbb{Z}_3) \times (\mathbb{Z}_2 \times \mathbb{Z}_7)$. Now sort the columns vertically:

$$\begin{array}{ccccccccl} 2^1 & \times & 3^0 & \times & 5^0 & \times & 7^0 & \times & \cdots \\ 2^1 & \times & 3^1 & \times & 5^0 & \times & 7^0 & \times & \cdots \\ 2^3 & \times & 3^3 & \times & 5^0 & \times & 7^1 & \times & \cdots \end{array}$$

Then reassemble:

$$\begin{array}{rcccccccccl} 2 & = & 2^1 & \times & 3^0 & \times & 5^0 & \times & 7^0 & \times & \cdots \\ 6 & = & 2^1 & \times & 3^1 & \times & 5^0 & \times & 7^0 & \times & \cdots \\ 1512 & = & 2^3 & \times & 3^3 & \times & 5^0 & \times & 7^1 & \times & \cdots \end{array}$$

so $\mathbb{Z}_{54} \times \mathbb{Z}_{24} \times \mathbb{Z}_{14} \cong \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{1512}$.