COMPLEX CHARACTER THEORY OF FINITE GROUPS

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Throughout, G is a finite group acting on a finite-dimensional complex vector space V, i.e. bearing a group homomorphism $G \to GL(V)$. We write the action as $g\vec{v}$, i.e. just using juxtaposition, except in special circumstances. I'll assume the language of group actions (on sets).

1. Reduction to irreps

A **subrepresentation** (or **subrep**) $W \leq V$ is a linear subspace invariant under the action, i.e. $g\vec{w} \in W \; \forall g \in G, \vec{w} \in W$.

A representation V is **reducible** if it has a subrep $W \leq V$ other than $0, W$. It is **irreducible** (or an **irrep**) if it is nonzero and not reducible. (So the 0 representation is neither.)

One obvious subrepresentation is the invariant vectors $V^G := \{\vec{v} \in V \; : \; g\vec{v} = \vec{v}, \; \forall g \in G\}$.

Hereafter we use the notation $\int_G$ as shorthand for $\frac{1}{|G|}\sum_{g\in G}$. Or, if you know how to do a left-invariant integral on your group and get $\int_G 1 = 1$, then maybe you can work with some infinite groups, like the circle.

Theorem 1.1. *There exists a Hermitian (sesquilinear positive definite) form $\langle , \rangle$ on V invariant under the G-action.*

Proof. Let $(,)$ be any Hermitian form, i.e. pick a basis and declare it to be orthonormal. Then define

$$\langle \vec{v}, \vec{w} \rangle := \int_G (g\vec{v}, g\vec{w})$$

and observe that this is sesquilinear, still positive definite, and now G-invariant. □

Corollary 1.2. *Every representation is a direct sum of irreps.*

Proof. If V is irreducible, we're done. Else it has a nontrivial subrep W. Pick an invariant Hermitian form $\langle , \rangle$ on V, and check that $W^\perp$ is also G-invariant. By assumption, $W, W^\perp$ are both lower-dimensional than V, so we can use induction. □

1.1. More about V^G.

Let $g|_V \in GL(V)$ denote the matrix associated to an acting element.

Lemma 1.3. *The operator $\pi_V := \int_G g|_V$ is a G-equivariant projection $V \twoheadrightarrow V^G$, with trace $\dim(V^G)$.*

Proof. It's easy to check that $\pi_V^2 = \pi_V$, hence is a projection onto its image. Also, $g\pi_V = \pi_V$, hence that image lies inside V^G. But obviously $\pi_V(\vec{v}) = \vec{v}$ for $\vec{v} \in V^G$, so the image is V^G. If we pick bases of V^G and $\ker \pi_V$ and concatenate them to a basis of V, then we can easily calculate the trace. □

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2. Schur's lemma

Given representations V of G and W of H, the vector space $\mathrm{Hom}(V, W) \cong V^* \otimes W$ is naturally a $(G \times H)$-representation, by $(g, h) \cdot \phi := h \circ \phi \circ g^{-1}$. If $H = G$, then we can restrict this representation to the diagonal $G \hookrightarrow G \times G$, $g \mapsto (g, g)$. The invariant vectors $\mathrm{Hom}(V, W)^G$ are written as $\mathrm{Hom}_G(V, W)$ and called **intertwining operators**; they are exactly the equivariant linear maps.

Lemma 2.1 (Schur)**.** *Let V, W be irreps. Then $\dim \mathrm{Hom}_G(V, W) = 1$ if $V \cong W$, 0 if $V \ncong W$.*

Proof. If $\phi \in \mathrm{Hom}_G(V, W)$ is not zero, then its kernel is not all of V (and is a subrepresentation, hence 0), and its image is not 0 (and is a subrepresentation, hence all of W), giving an isomorphism.

If $\rho \in \mathrm{Hom}_G(V, W)$, then $\phi^{-1} \circ \rho \in \mathrm{Hom}_G(V, V)$, and must have an eigenvalue λ. Hence $(\phi^{-1} \circ \rho) - \lambda$ is an equivariant map with a nonzero kernel, which must be V. So $(\phi^{-1} \circ \rho) - \lambda = 0$, i.e. ρ is a multiple of ϕ. □

Theorem 2.2 (Jordan-Hölder for representations)**.** *Given two decompositions of V into irreps, the number of times an irrep W occurs in the decompositions is the same.*

Proof. That number is $\dim \mathrm{Hom}_G(W, V)$, by composing the projections of V to its pieces with the maps from W, and applying Schur's lemma. □

Let $\mathrm{Rep}(G)$ denote the ring of formal differences of representations of G, with multiplication and addition derived from direct sum and tensor product. Given two representations V, W and associated elements $[V], [W] \in \mathrm{Rep}(G)$, define

$$\langle [V], [W] \rangle := \dim \mathrm{Hom}_G(V, W)$$

as an inner product on $\mathrm{Rep}(G)$. (One must extend linearly in order to define it on formal differences.)

Corollary 2.3. *This inner product is well-defined, symmetric, and positive definite. The irreps are an orthonormal basis for $\mathrm{Rep}(G)$.*

Proof. The irreps span by corollary 1.2, are independent by theorem 2.2, and are orthonormal by Schur's lemma. One can check well-definedness by expanding everybody into irreps and counting them. □

Theorem 2.4. *Let V be an irrep of $G \times H$. Then $V \cong A \otimes B$ for two irreps A, B of G, H.*

Proof. Pick an H-irrep B inside V (forgetting the G-action), and think of it as a $(G \times H)$-rep by giving it the trivial G-action. Define $A := \mathrm{Hom}_H(B, V) = \mathrm{Hom}(B, V)^{1 \times H}$, where $\mathrm{Hom}(B, V)$ is a $(G \times H)$-rep since B, V each are $(G \times H)$-reps. Then since $G \times 1$ normalizes $1 \times H$, it acts on A, making A a G-rep. Note that one element of A is the inclusion map $B \hookrightarrow V$.

Now we have a map $A \otimes B \to V$ given by $\phi \otimes \vec{b} = \phi(\vec{b})$, easily checked to be $G \times H$-equivariant, and nonzero if we take ϕ to be that inclusion map. Hence its image is a nonzero subrep of V, so is all of V by irreducibility.

Finally, we check that this map is $1:1$. As a map of H-reps, this sticks a number of copies of B into V, and since it's onto we learn V is H-isomorphic to B^k for some k. (I.e.

we split $A \otimes B$ into the kernel plus a complement, and use theorem 2.2 to know that complement is a sum of Bs.) By Schur's lemma, $\dim A = k$. Hence $\dim V = \dim(A \otimes B)$, so the onto map must be 1 : 1. □

3. Characters

From lemma 1.3,

$$\dim \mathrm{Hom}_G(V, W) \quad = \quad \dim \mathrm{Hom}(V, W)^G = \mathrm{trace}(\pi_{\mathrm{Hom}(V,W)}) = \int_G \mathrm{trace}(g|_{\mathrm{Hom}(V,W)})$$

Lemma 3.1. (1) *If* $V \cong W$*, then* $\mathrm{trace}(g|_V) = \mathrm{trace}(g|_W)$.
(2) $\mathrm{trace}(g|_{V\otimes W}) = \mathrm{trace}(g|_V)\mathrm{trace}(g|_W)$
(3) $\mathrm{trace}(g|_{V^*}) = \overline{\mathrm{trace}(g|_V)}$

Proof. (1) Picking bases, $g|_V$ and $g|_W$ become matrices conjugate by the matrix $\phi : V \to W$, and hence have the same trace.
(2) Since g acts unitarizably on V and W, it acts diagonalizably. Pick bases w.r.t. which the actions are by diagonal matrices. Then the action on the tensor product is given by the Kronecker product of the two matrices, whose diagonal entries are all the pairwise products, whose sum is therefore the product of the two sums.
(3) The action on V^* comes from g^{-1}, and $\mathrm{trace}(M^{-1}) = \overline{\mathrm{trace}(M)}$ for unitary matrices (since their eigenvalues are of norm 1).

□

Consequently,

$$\int_G \mathrm{trace}(g|_{\mathrm{Hom}(V,W)}) = \int_G \mathrm{trace}(g|_{V^*\otimes W}) = \int_G \mathrm{trace}(g|_{V^*})\,\mathrm{trace}(g|_W) = \int_G \overline{\mathrm{trace}(g|_V)}\,\mathrm{trace}(g|_W)$$

This motivates the definition of the **character** χ_V of a representation:

$$\chi_V(g) := \mathrm{trace}(g|_V)$$

Unwrapping the g from the notation, χ_V lies in $\mathrm{Fun}(G)$, the space of complex-valued functions on G. We will make this into a ring via pointwise addition and multiplication (i.e. from $\mathbb{C}$'s ring structure).

Lemma 3.2. $\chi_{V\oplus W} = \chi_V + \chi_W$

Proof. In matrix terms, $g|_{V\oplus W}$ is block diagonal with $g|_V$, $g|_W$ as its blocks, so the trace is additive. □

On $\mathrm{Fun}(G)$ we define a Hermitian inner product

$$\langle a, b\rangle := \int_G \overline{a(g)}b(g)$$

and unwrap one more level:

Theorem 3.3. χ *descends to a map* $\mathrm{Rep}(G) \to \mathrm{Fun}(G)$*, a ring homomorphism, and an isometry w.r.t. the two inner products. In particular it is injective.*

Proof. Lemmata 3.1 and 3.2 establish the ring homomorphism statement. Separately, we matched up the two inner products. If a vector is in the kernel, it has norm square zero in the target, hence norm square zero in the source, so by positive definiteness must be zero. □

In particular, the number of distinct irreps is finite!

The map obviously isn't onto – the source is a lattice $\mathbb{Z}^{\#\text{irreps}}$, and the target a complex vector space – but it doesn't even usually span the target. Let $\mathrm{ClFun}(G)$ denote the functions that are constant on conjugacy classes, and observe that $\chi_V \in \mathrm{ClFun}(G)$ by the fact that $\mathrm{trace}(XYX^{-1}) = \mathrm{trace}(Y)$.

4. The group algebra

To show that $\mathrm{Rep}(G)$ does indeed span $\mathrm{ClFun}(G)$, we need enough irreps of G, and we'll look for them inside

$$\mathbb{C}[G] := \{ \text{ formal sums } \sum_g \lambda_g g \}$$

which later we'll consider as the **group algebra**, by linearly extending the multiplication from the group. (So unlike the formally similar $\mathrm{Fun}(G)$, this is usually not commutative.)

Theorem 4.1 (Peter-Weyl[1]). *As a $G \times G$-representation, $\mathbb{C}[G] \cong \bigoplus_V (V^* \otimes V)$, where V runs over the irreps.*

Proof. It's some sum of $G \times G$-irreps, which we classified in theorem 2.4, so we use orthonomality to find out which ones. □

Proof.

$$\begin{aligned}
\langle \chi_{\mathbb{C}[G]}, \chi_{V\otimes W} \rangle &= \int_{G\times G} \overline{\mathrm{trace}((g,h)|_{\mathbb{C}[G]})}\, \chi_{V\otimes W}(g,h) = \int_{G\times G} \sum_{k\in G} [gkh^{-1} = k]\, \chi_V(g)\, \chi_W(h) \\
&= \frac{1}{|G|^2} \sum_{g\in G} \chi_V(g) \sum_{k\in G} \sum_{h\in G} [k^{-1}gk = h]\, \chi_W(h) \qquad \text{where } [\mathrm{true}] = 1, [\mathrm{false}] = 0 \\
&= \frac{1}{|G|^2} \sum_{g\in G} \chi_V(g) \sum_{k\in G} \chi_W(k^{-1}gk) = \frac{1}{|G|^2} \sum_{g\in G} \chi_V(g) \sum_{k\in G} \chi_W(g) \\
&= \frac{1}{|G|^2} \sum_{g\in G} \chi_V(g)\chi_W(g)\, |G| = \langle \chi_{V^*}, \chi_W \rangle
\end{aligned}$$

which is 1 if $W \cong V^*$ and 0 otherwise by Schur's lemma, so $V \otimes W$ occurs in $\mathbb{C}[G]$ exactly once in that case. □

Corollary 4.2. *The number of irreps is the number of conjugacy classes, hence the characters of irreps are an (orthonormal) basis of the ring $\mathrm{ClFun}(G)$ of class functions.*

[1]They actually work with $L^2(G)$ where G is a compact group, and $\oplus$ is the L^2 direct sum, so it's rather more impressive.

Proof. We consider the G-invariants in $\mathbb{C}[G]$. Each summand $V^* \otimes V$ has a 1-dimensional invariant space, by Schur's lemma, giving a basis to the invariant space. But it's easy to see that the invariant space is linear combinations whose coefficients are constant on conjugacy classes, i.e. the conjugacy classes give a basis of the dual space. □

Corollary 4.3. $|G| = \sum_V (\dim V)^2$.

If G acts on V, then we can linearly extend the map $G \to GL(V) \to End(V)$ to an algebra map $\mathbb{C}[G] \to End(V)$, and it is $G \times G$-equivariant. By the irreducibility of $End(V) \cong V^* \otimes V$ as a $G \times G$-representation, this map must be onto.

Theorem 4.4 (Artin-Wedderburn, for group algebras)**.** *As an algebra,* $\mathbb{C}[G] \cong \bigoplus_V End(V)$.

Proof. We add up the maps to the individual $End(V)$. For each one, the kernel is $\oplus_{W \neq V} End(W)$, and the intersection of these kernels is zero. Hence the map is 1 : 1, and the two algebras are the same dimension, $|G|$. □

This is where the idempotent $\frac{1}{|G|} \sum_{g \in G} g$ *really* lives, whose image π_V we considered in $End(V)$ for each rep V. The other idempotents are also easy to construct:

$$\pi^V := \frac{\dim V}{|G|} \sum_{g \in G} \overline{\chi_V(g)} g$$

Exercise: Show that this projects any representation to its V-isotypic component, i.e. the sum of the irreducible subreps isomorphic to V.

5. Character tables

Given an ordering on the conjugacy classes (which will be the columns) and the irreducible characters (which will be the rows), we make a square (!) matrix with entries $\chi_V(g)$, called a **character table**.

Theorem 5.1. *If one multiplies column* $[g]$ *by* $|C_G(g)|^{-1/2}$*, the resulting matrix is unitary. Hence the columns of a character table are orthogonal, and the norm-square of the* $[g]$ *column is* $|C_G(g)|$.

Proof. The rows are now orthonormal by the orthonormality of characters, and the fact that $|C_G(g)|^{-1} = |[g]|/|G|$. Hence the columns are now orthonormal. Unscaling the columns, we get the claimed results. □

This includes corollary 4.3 where $g = e$, traditionally the leftmost column of the character table. Likewise, one puts the trivial character $\chi_V \equiv 1$ in the first row.