HOMOLOGICAL ALGEBRA

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CONTENTS

Fix a commutative base ring k (the minimal choice being $k = \mathbb{Z}$). Then a k**-algebra** R is a ring with unit, plus a map $k \to Z(R)$. In particular the biadditive multiplication $R \times R \to R$ factors through the tensor product $R \otimes_k R \to R$. Hereafter $\otimes$ will denote $\otimes_k$.

1. LEFT AND RIGHT MODULES

Unless otherwise noted, R is a noncommutative ring with unit, and we indicate left and right R-modules by ${}_RM$, M_R. We can make any left R-module into a right R^{op}-module. The least structure our objects will have is being a k-module, i.e. being an abelian group if $k = \mathbb{Z}$.

An (R, S)**-bimodule** ${}_RM_S$ possesses two commuting actions, or equivalently, is a module over $R \otimes S^{op}$. As such any left R-module is naturally a (R, k)-bimodule.

1.1. Hom.
There is natural *left* S-module structure on $\mathrm{Hom}_R({}_RM_S, {}_RN)$:

$$(s \cdot f)(m) := f(ms)$$

First we check that $s \cdot f$ is again in $\mathrm{Hom}_R(M, N)$:

$$(s \cdot f)(rm) = f(rms) = r\, f(ms) = r\,(s \cdot f)(m)$$

Then that the action of S is a left action:

$$(s \cdot t \cdot f)(m) = (t \cdot f)(ms) = f(mst) = (st \cdot f)(m)$$

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In particular, if M_S is only a right S-module, we write $M^* := \mathrm{Hom}_k(M, k)$ and get a left S-module. The natural map $M \to M^{**}$ is not usually $1:1$ or onto.

The mnemonic, for when k is a field, is that $\mathrm{Hom}_R(M, N)$ is something like $M^* \otimes N$, and the dual of a right module is a left module.

Of course we're used to having inner products on vector spaces, and now that looks impossible. So we ask for an **antiautomorphism** $^-: R \to R$ i.e. an isomorphism $R \to R^{op}$. Then we can make M^* into a left R-module, and consider homomorphisms $M \to M^*$. If we want the natural function $M \to M^{**}$ to be R-linear, then $^-$ should be an antiïnvolution. Of course the most familiar case is R commutative and $^-$ the identity.

Fun case: $k = \mathbb{C}$, $^- =$ complex conjugation, and then these homomorphisms give sesquilinear forms. This extends to the quaternionic case in a way that $^- = \mathrm{Id}$ doesn't. Another is $g \mapsto g^{-1}$ for $k[G]$.

If $N = {}_R N_T$ is an (R, T)-bimodule, then the hom-space is also a right module.

If R is commutative, we can soup up any ${}_R M$ to ${}_R M_R$, hence $\mathrm{Hom}_R(M, N)$ is again a (left or right) R-module. But if we try to use an antiinvolution, the actions of R and R^{op} may not commute, so may not make M into a bimodule.

1.2. $\otimes$**.** The most general natural tensor is ${}_R M_S \otimes_S {}_S N_T$, which is then a (R, T)-bimodule. Of course a common case is to start with R commutative and ${}_R M, {}_R N$.

A fun 2-category: rings, bimodules, bimodule homomorphisms.

1.3. **Adjointness.**

Theorem 1.1. *Let ${}_R M_S$ be a bimodule, so we have functors*

$$M \otimes_S \bullet : \quad \mathrm{Mod}_S \quad \longleftrightarrow \quad \mathrm{Mod}_R \quad : \mathrm{Hom}_R(M, \bullet)$$

Then these are adjoint.

Proof. Let A, B be left R-, S-modules respectively. Then we need to show

$$\begin{aligned} \mathrm{Hom}_S(B, \mathrm{Hom}_R(M, A)) &\cong \mathrm{Hom}_R(M \otimes B, A) \\ f &\mapsto \Big(m \otimes b \mapsto f(b)\Big(m\Big)\Big) \end{aligned}$$

is bijective, and natural in A, B. ∎

2. Exact functors

An **additive category** is a category object in the monoidal category of abelian groups, i.e. the homsets are abelian groups and composition is biadditive. We'll actually want a better thing, that all homsets are k-modules.

Let $0 \to A \to B \to C \to 0$ be a short exact sequence of some kind of modules, and T an **additive functor** from their module category to another (again respecting the additive structure – so e.g. not Schur functors). Then T is **left exact** resp. **right exact** if $0 \to T(A) \to T(B) \to T(C) \to 0$ is exact at $T(A), T(B)$ resp. $T(B), T(C)$, and **exact** if both.

Proposition 2.1. *An exact functor preserves exactness of arbitrary exact sequences (not just short exact).*

Proof. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be part of an exact sequence, and T an exact functor. We can factor it as $A \twoheadrightarrow A' \hookrightarrow B \twoheadrightarrow C' \hookrightarrow C$, where $A' \hookrightarrow B \twoheadrightarrow C'$ is short exact. Apply T to that sequence to get

$$T(A) \twoheadrightarrow T(A') \hookrightarrow T(B) \twoheadrightarrow T(C') \hookrightarrow T(C)$$

since T preserves $1:1$ness and ontoness. Now the kernel of $T(B) \to T(C)$ is the kernel of $T(B) \to T(C')$, which is the image of $T(A') \to T(B)$ by exactness at B. But $T(A) \twoheadrightarrow T(A')$ says that it's the same as the image of $T(A) \to T(B)$. □

Lemma 2.2. $\mathrm{Hom}(Z, \bullet)$ *is left exact.*

Proof. We apply $\mathrm{Hom}(Z, \bullet)$ to $0 \to A \hookrightarrow B \to C$, or really to $0 \to A \hookrightarrow B \to B/A$, obtaining

$$\mathrm{Hom}(Z, A) \to \mathrm{Hom}(Z, B) \to \mathrm{Hom}(Z, B/A)$$

Since $A \leq B$, the first map is $1:1$. If $\phi : Z \to B$ is in the kernel of the second map, that means $Z \to B \to B/A$ is zero, i.e. the image of Z lies in A. So we obtain an element of $\mathrm{Hom}(Z, A)$. □

The main examples are $\mathrm{Hom}_R(M, \bullet)$ and $M \otimes_S \bullet$, as follows from the next theorem. We need the definition of an object $X \in \mathrm{Obj}(\mathcal{C})$ **representing** a covariant functor $F : \mathcal{C} \to \mathbf{Set}$, namely $F \cong \mathrm{Hom}_{\mathcal{C}}(X, \bullet)$.

Theorem 2.3. *Let* $\mathcal{C}$ *have an object* R *that represents the forgetful functor. Then if* $L : \mathcal{C} \leftrightarrow \mathcal{D} : R$ *are adjoint functors, then the left adjoint is right exact and vice versa.*

Proof. We have bijections $\mathrm{Hom}(X_1, Z) \cong \mathrm{Hom}(X_2, Z)$ that are natural in Z. For $Z = X_1$ the left side has the identity, giving us an element of $\mathrm{Hom}(X_2, X_1)$, and for $Z = X_2$ the RHS does, giving us an element of $\mathrm{Hom}(X_2, X_1)$; these turn out to be inverses. □

Corollary 2.4. *If* $\mathrm{Hom}(Z, A) \to \mathrm{Hom}(Z, B) \to \mathrm{Hom}(Z, C)$ *is exact for all* Z*, then* $A \to B \to C$ *was exact.*

Proof that adjointness $\Longrightarrow$ *exactness.* Start with an exact sequence $0 \to A \to B \to C$ in $\mathcal{D}$ (the $\to 0$ is unimportant). Apply R to get

$$0 \to R(A) \to R(B) \to R(C)$$

in $\mathcal{C}$. To make use of adjointness, we need to map into this, so $\mathrm{Hom}_{\mathcal{C}}$ out of Z to get

$$0 \to \mathrm{Hom}_{\mathcal{C}}(Z, R(A)) \to \mathrm{Hom}_{\mathcal{C}}(Z, R(B)) \to \mathrm{Hom}_{\mathcal{C}}(Z, R(C))$$

Adjointness says that we have a commuting diagram

$$\begin{array}{ccccccc} 0 & \to & \mathrm{Hom}_{\mathcal{D}}(L(Z), A) & \to & \mathrm{Hom}_{\mathcal{D}}(L(Z), B) & \to & \mathrm{Hom}_{\mathcal{D}}(L(Z), C) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathrm{Hom}_{\mathcal{C}}(Z, R(A)) & \to & \mathrm{Hom}_{\mathcal{C}}(Z, R(B)) & \to & \mathrm{Hom}_{\mathcal{C}}(Z, R(C)) \end{array}$$

with vertical isomorphisms.

We already proved that the top sequence is exact, hence the bottom one is, for all Z. Taking the case $Z = R$, we get $0 \to A \to B \to C$ is exact. □

In fact using the Yoneda lemma,

Lemma 2.5 (Yoneda). *If two objects represent the same functor, they are canonically isomorphic.*

one can dispense with the object representing the forgetful functor.

3. Projective and injective modules

Even though $\mathrm{Hom}(M, \bullet)$ isn't left-exact for all ${}_R M$, it is for some, e.g. for R itself, and more generally for free modules. Similarly $M \otimes \bullet$ is exact for M free.

However, free is a difficult property to maintain. One result we have in this direction derives from

Theorem 3.1. *Let M be a finitely generated module over a PID R. Then M is a direct sum of cyclic modules $R/r_i R$.*

namely

Corollary 3.2. *If R is a PID, then a finitely generated submodule of R^n is free.*

but more generally, we will work with "projective" modules P: if given $M \twoheadrightarrow N$ and $P \to N$, there always exists a lift $P \dashrightarrow M$, then P is **projective**.

Theorem 3.3.
(1) *Free modules are projective.*
(2) *If $P = Q \oplus R$, then the* **direct summands** *Q, R are projective.* *(Even this already is false for free modules: try $M = R = \mathbb{Z}/6$.)*
(3) *Any SES $0 \to A \to B \to P \to 0$ with P projective, splits.* *I.e. any map $B \twoheadrightarrow P$ must be a* **projection** *$A \times P \twoheadrightarrow P$.*
(4) *P is projective iff it's a direct summand of a free module.*
(5) *If every SES onto P splits, then P is projective.*

Proof.
(1) The generators of P go somewhere in N. By the ontoness, we can lift those to M, obtaining a map $P \to M$.
(2) Given a map $Q \to N$ we can extend it to P (by 0 on R), lift to $P \to M$, restrict to $Q \to M$.
(3) Take $N = P$ i.e. $P \to N$ the identity. Lift to get a map $P \to B$. Now check that B is the sum of the images of A and P.
(4) $\Longrightarrow$ Pick generators of P to get a surjection $F \twoheadrightarrow P$. Now apply the previous to $0 \to \ker \to F \to P \to 0$. $\Longleftarrow$ (2).
(5) If every map $B \twoheadrightarrow P$ splits, then take B free, and use (4).

∎

Another reason people like projectivity is that it can be checked "locally", i.e. after M is a projective R-module iff $R[a^{-1}] \otimes_R M$ is a projective $R[a^{-1}]$-module for enough a, but the same statement doesn't work with "free".

A **free resolution** of a module M is an exact sequence

$$\cdots \to F_3 \to F_2 \to F_1 \to F_0 \twoheadrightarrow M \to 0$$

where the F_i are free; we could take F_0 to be freely generated by M, then F_1 freely by the kernel of $F_0 \twoheadrightarrow M$, and so on.

Example. Let $R = \mathbb{C}[a, b, c]/\langle b^2 - ac\rangle$, and $M = R/\langle a, b\rangle$. Then we have a free resolution

$$\cdots R^2 \xrightarrow{[c]\oplus[-b]} R^2 \xrightarrow{[a]\oplus[b]} R^2 \xrightarrow{[c]\oplus[-b]} R^2 \overset{[a\ b]}{\twoheadrightarrow} M \to 0$$

Theorem 3.4 (Hilbert basis theorem)**.** *If* M *is a module over* $\mathbb{F}[x_1, \ldots, x_n]$*, then* M *has a free resolution with* $F_i = 0 \ \forall i > n$.

Proof. Use a Gröbner basis for the module, and... □

More generally though, we'll use **projective resolutions**, which look the same but F_i need only be projective.

3.1. **Injective modules.** All these notions can be dualized by reversing the arrows, but they can get kind of weird. What would an injective $\mathbb{Z}$-module I be? Given any $M \leq M'$, and $M \to I$, there should be an extension $M' \to I$.

A $\mathbb{Z}$-module M is **divisible** if for every $m \in M$ and $a \nmid 0$ there exists m' with $am' = m$. The principal example is $\mathbb{Q}/\mathbb{Z}$ as a $\mathbb{Z}$-module.

Theorem 3.5.
(1) *Injective modules are divisible.*
(2) *If* M *is torsion-free and divisible, it is injective.*

Proof.
(1) Consider the maps $R \overset{a\cdot}{\hookleftarrow} R \overset{\cdot m}{\longrightarrow} M$, where the first is injective by $a \nmid 0$. Then injectivity says that this should complete to $R \to M$, with $1 \mapsto m'$.
(2) ... this is a real pain, and uses Zorn's lemma.

□

In particular, it's much less obvious (but true) that ${}_R\mathbf{Mod}$ has enough injectives to make an injective resolution of any module.

4. Complexes and homology

A **complex** $A_n \overset{\phi_n}{\to} A_{n-1} \overset{\phi_{n-1}}{\to} \cdots \overset{\phi_1}{\to} A_0$ is a sequence of maps, the **differentials**, in an additive category such that all composites $\phi_i \circ \phi_{i+1} = 0$. Then (assuming the category has quotients, as in module categories) we can define the **homology** as $H_i := \ker \phi_i / image(\phi_{i+1})$. Under this definition, an exact sequence is a complex with vanishing homology.

There are obvious definitions of direct sums, subcomplexes, and isomorphisms of complexes (a sequence of maps giving commuting squares). The two basic kinds of complexes are complexes concentrated in a single degree (with trivial differential, thus isomorphic to their homology) and two-step complexes $\cdots \to 0 \to k \to k \to 0 \to \cdots$ with no homology.

Theorem 4.1. *Let* (A_i) *be a finite complex of finite-dimensional vector spaces over* k*. Then* (A_i) *is isomorphic to the direct sum of its homology (with trivial differential) and a bunch of two-step complexes.*

Proof. If the last map is not onto, split A_0 into $image(\phi_1)$ and a complement, then use induction on total dimension of the complex.

Now say it is onto. Consider the subspaces $image(\phi_2) \leq \ker(\phi_1) \leq A_1$. Pick complements H and T, and split $\to A_1 \to A_0 \to 0$ into

$$\begin{array}{ccccccccc} & (\to & A_2 & \twoheadrightarrow & image(\phi_2) & \to & 0 & \to & 0) \\ \oplus & (\to & 0 & \to & H & \to & 0 & \to & 0) \\ \oplus & (\to & 0 & \to & T & \cong & A_1 & \to & 0) \end{array}$$

and use induction on the length of the complex. □

Very often we will only care about the homology of a complex, because the complex involves choices but the homology does not. So we want a way to prove that two complexes have the same homology. Before defining isomorphisms, we define a **morphism of complexes** as a diagram

$$\begin{array}{ccccccc} \cdots & A_2 & \to & A_1 & \to & A_0 & \cdots \\ & \downarrow & = & \downarrow & = & \downarrow & \\ \cdots & B_2 & \to & B_1 & \to & B_0 & \cdots \end{array}$$

Theorem 4.2. *A morphism of complexes induces a map on homology.*

Proof. The map $A_1 \to B_1$ takes $\mathrm{image}(A_2 \to A_1)$ to $\mathrm{image}(A_2 \to B_1)$ through B_2 by the left square, so lands inside $\mathrm{image}(B_2 \to B_1)$.

It's only well-defined up to $\ker(A_1 \to A_0)$, which lies inside $\ker(A_1 \to B_0)$, which maps to $\ker(B_1 \to B_0)$. So as long as we kill that in the target, we get a well-defined map $H_1(A) \to H_1(B)$. □

We may as well do this here:

Theorem 4.3. *A SES of complexes induces a long exact sequence on homology.*

Proof.

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots & A_2 & \overset{a_1}{\to} & A_1 & \overset{a_0}{\to} & A_0 & \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots & B_2 & \overset{b_1}{\to} & B_1 & \overset{b_0}{\to} & B_0 & \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots & C_2 & \overset{c_1}{\to} & C_1 & \overset{c_0}{\to} & C_0 & \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array}$$

We already have the maps $H_i(A) \to H_i(B) \to H_i(C)$, and need the map $H_i(C) \to H_{i-1}(A)$, say $i = 1$ in the above.

Start with $c_1 \in C_1 \mapsto 0 \in C_0$. Then $B_1 \twoheadrightarrow C_1$ so $\exists b_1 \mapsto c_1$, mapping to $b_0 \in B_0$. Then by the commuting square, $b_0 \mapsto 0 \in C_0$. Hence it the image of a unique $a_0 \in A_0$.

What if the original c_1 were the image of some $c_2 \in C_2$? Then we could lift that c_2 to b_2, whose image $b_1' \in B_1$ would go the same place as b_1 does (namely c_1). Hence $b_1 = b_1' + a_1$ for some unique a_1.

Now $a_1 \mapsto b_1 - b_1' \mapsto b_0$, since $b_2 \mapsto b_1' \mapsto 0$ horizontally. But a_0 is the unique element mapping to b_0, so a_0 must be the image of a_1. I.e. if we lift image elements, we land inside image elements, giving a well-defined map on homology. □

We would like a natural reason that two complexes $(A_i), (B_i)$ would have the same homology. More generally, we would like a natural reason that two maps $f, g : (A_i) \to (B_i)$ would induce the same map on homology. Define a **chain homotopy** from f to g to be a collection of maps $h_i : A_i \to B_{i+1}$ (*not* a map of complexes) such that $f - g = d_B h + h d_A$.

(The RHS is something like a commutator; note that the d and h operations are both odd degree, so it is natural to have a $+$ instead of a $-$ in their "commutator").

Proposition 4.4. *If* $f, g : (A_i) \to (B_i)$ *have a chain homotopy* h *relating them, then they induce the same map on homology.*

Proof. Let $a_i \in \ker(A_i \to A_{i-1})$. We need to show that $\mathrm{image}(B_{i+1} \to B_i) \ni (f-g)(a_i) = (d_B h + h d_A)(a_i) = d_B(h(a_i)) + h(d_A(a_i)) = d_B(h(a_i)) + 0$. □

In topology, a natural source of such hs is from maps $[0,1] \times \mathcal{A} \to \mathcal{B}$.

Sometimes we have maps $(A_i) \to (B_i) \to (A_i)$, where one composite is the identity and the other composite is only chain homotopic to the identity, e.g. in topology from the inclusion of a deformation retract. Then the above says that the homologies are isomorphic.

5. Mapping resolutions

Say we have a map $M \to N$ and resolutions (i.e. exact complexes) of M, N:

$$\begin{array}{ccccccccccc} \cdots & A_2 & \xrightarrow{a_1} & A_1 & \xrightarrow{a_0} & A_0 & \twoheadrightarrow & M & \to & 0 \\ & & & & & & & \downarrow & & \\ \cdots & B_2 & \xrightarrow{b_1} & B_1 & \xrightarrow{b_0} & B_0 & \twoheadrightarrow & N & \to & 0 \end{array}$$

Proposition 5.1. *If* (A_i) *is a projective resolution, then we can extend the above to a map of complexes, uniquely up to chain homotopy.*

Proof. Since $B_0 \twoheadrightarrow N$ and $A_0 \to M \to N$ and A_0 projective, there exists a map $f : A_0 \to B_0$ making a commuting square.

We can obviously replace A_0 with $\mathrm{image}(A_1 \to A_0)$ whose map to M is the zero map (since the top row is a complex), and replace M with 0, but we want to also replace B_0 with $\mathrm{image}(B_1 \to B_0)$ and N with 0. To do that safely, we need to know that $f(\mathrm{image}(A_1 \to A_0)) \leq \mathrm{image}(B_1 \to B_0)$.

If we already had the map $A_1 \to B_1$, we could use that. Instead we use

$$\mathrm{image}(A_1 \to A_0) = \ker(A_0 \to M) \leq \ker(A_0 \to N) \xrightarrow{f} \ker(B_0 \to N) = \mathrm{image}(B_1 \to B_0)$$

Now we can indeed rip off that last column and repeat the argument.

To see the uniqueness, let f, g be two such maps. Start with $h_{-1} := 0$ (the map $M \to B_0$). Assume we have $h_0, \ldots, h_{i-1}$ satisfying $f - g = d_B h + h d_A$, and now we want $h_i : A_i \to B_{i+1}$ solving $b_i h_i = g_i - f_i - h_{i-1} a_{i-1}$.

$$\begin{array}{ccccccccccc} \cdots & A_2 & \xrightarrow{a_1} & A_1 & \xrightarrow{a_0} & A_0 & \twoheadrightarrow & M & \to & 0 \\ & \downarrow\downarrow & & \downarrow\downarrow & \swarrow & \downarrow\downarrow & \swarrow & \downarrow & & \\ \cdots & B_2 & \xrightarrow{b_1} & B_1 & \xrightarrow{b_0} & B_0 & \twoheadrightarrow & N & \to & 0 \end{array}$$

Obviously we'll use projectivity to "construct" desired maps, which means we need to know that A_i maps via the RHS into the image of $B_{i+1} \to B_i = \ker(B_i \to B_{i-1})$. So we compute:

$$\begin{aligned} b_{i-1}(g_i - f_i - h_{i-1}a_{i-1}) &= b_{i-1}g_i - b_{i-1}f_i - b_{i-1}h_{i-1}a_{i-1} \\ &= b_{i-1}g_i - b_{i-1}f_i - (g_{i-1} - f_{i-1} - h_{i-2}a_{i-2})a_{i-1} \\ &= b_{i-1}g_i - b_{i-1}f_i - g_{i-1}a_{i-1} + f_{i-1}a_{i-1} \\ &= (b_{i-1}g_i - g_{i-1}a_{i-1}) - (b_{i-1}f_i - f_{i-1}a_{i-1}) \\ &= 0 - 0 = 0 \end{aligned}$$

□

6. DERIVED FUNCTORS

Let $(A_i), (B_i)$ two resolutions of the same object M. Then we get maps between the resolutions, unique up to chain homotopies

$$\begin{array}{ccccccccc} \cdots & A_2 & \stackrel{a_1}{\to} & A_1 & \stackrel{a_0}{\to} & A_0 & \twoheadrightarrow & M & \to & 0 \\ & \downarrow\downarrow & \swarrow & \downarrow\downarrow & \swarrow & \downarrow\downarrow & \swarrow & \| & & \\ \cdots & B_2 & \stackrel{b_1}{\to} & B_1 & \stackrel{b_0}{\to} & B_0 & \twoheadrightarrow & M & \to & 0 \\ & \downarrow\downarrow & \swarrow & \downarrow\downarrow & \swarrow & \downarrow\downarrow & \swarrow & \| & & \\ \cdots & A_2 & \stackrel{a_1}{\to} & A_1 & \stackrel{a_0}{\to} & A_0 & \twoheadrightarrow & M & \to & 0 \end{array}$$

which would induce maps on homologies, except that's silly. It's worth noting that the composite maps and identity maps must be chain homotopic to one another (which would induce isomorphism of homologies, except that's sillier).

Now ditch the M, and apply a covariant right exact functor $T : \mathcal{C} \to \mathcal{D}$ (soon, $X \otimes \bullet$) everywhere:

$$\begin{array}{ccccccccc} \cdots & T(A_2) & \stackrel{T(a_1)}{\to} & T(A_1) & \stackrel{T(a_0)}{\to} & T(A_0) & \to & 0 \\ & \downarrow\downarrow & \swarrow & \downarrow\downarrow & \swarrow & \downarrow\downarrow & & \\ \cdots & T(B_2) & \stackrel{T(b_1)}{\to} & T(B_1) & \stackrel{T(b_0)}{\to} & T(B_0) & \to & 0 \\ & \downarrow\downarrow & \swarrow & \downarrow\downarrow & \swarrow & \downarrow\downarrow & & \\ \cdots & T(A_2) & \stackrel{T(a_1)}{\to} & T(A_1) & \stackrel{T(a_0)}{\to} & T(A_0) & \to & 0 \end{array}$$

The result is two complexes – no longer exact – with morphisms between them, each composite chain homotopic to the identity. Hence their homologies are isomorphic.

To get an actual canonical definition, we can use the canonical free resolution of M (use every element to generate, etc.), even though it's enormous. (We'll just never compute that way.) Given

$$\begin{array}{ccccccccc} \cdots & A_2 & \stackrel{a_1}{\to} & A_1 & \stackrel{a_0}{\to} & k^M & \twoheadrightarrow & M & \to & 0 \\ & & & & & & & \downarrow & & \\ \cdots & B_2 & \stackrel{b_1}{\to} & B_1 & \stackrel{b_0}{\to} & k^N & \twoheadrightarrow & N & \to & 0 \end{array}$$

the canonical free resolutions, we get a natural map $k^M \to k^N$. Then since that square commutes, the map takes $\ker(k^M \to M) \to \ker(k^N \to N)$, with which we can define the next map, and so on. This shows that the canonical free resolution is functorial, and from there, that the homology of the complex

$$\cdots \quad T(A_2) \stackrel{T(a_1)}{\to} T(A_1) \stackrel{T(a_0)}{\to} T(A_0) \to 0$$

is functorial in M. These are called the **left derived functors** $L_i T$ of the right exact functor T.

What does right exactness get us? The complex $T(A_1) \to T(A_0) \to T(M) \to 0$ is still exact at $T(A_0)$, i.e. $T(A_1) \to T(A_0) \to 0$ isn't; its homology is $T(M)$. So $L_0 T = T$.

Example. Let $T = \mathbb{Z}_n \otimes \bullet : Ab \to Ab$, where $\mathbf{Ab} = {}_{\mathbb{Z}}\mathbf{Mod}$ denotes finitely generated abelian groups. Let $g = gcd(m, n)$. Then we can resolve $\mathbb{Z}_n$ as the 0th homology of

$$\begin{array}{lccccccc} & 0 & \to & \mathbb{Z} & \stackrel{m\cdot}{\to} & \mathbb{Z} & \to & 0 \\ \text{whose } T \text{ is} & 0 & \to & \mathbb{Z}_n & \stackrel{m\cdot}{\to} & \mathbb{Z}_n & \to & 0 \\ \text{with kernels} & & & \mathbb{Z}_{n/g} & & \mathbb{Z}_n & & \\ \text{and images} & & & 0 & & \mathbb{Z}_{n/g} & & \\ \text{giving homology} & & & \mathbb{Z}_{n/g} & & \mathbb{Z}_g & & \end{array}$$

Define $Tor_i(M, N) := H_i(M \otimes$ a projective resolution of $N)$. With this definition, it's quite nonobvious that $Tor_i(M, N) \cong Tor_i(N, M)$.

The long exact sequence. Let T be right exact, and $0 \to A \to B \to C \to 0$ a SES. Then the covariant functors $L^i T$ give us complexes

$$0 \to (L^i T)(A) \to (L^i T)(B) \to (L^i T)(C) \to 0$$

but now we claim that there is a natural map $(L^i T)(C) \to (L^{i-1} T)(A)$ gluing them together into a long exact sequence.

We know how to get those from short exact sequences of complexes. The $L^i T$ come from T(projective resolution). So we want to replace $0 \to A \to B \to C \to 0$ by a short exact sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ of projective resolutions, then apply T, then get the LES. Such a SES would necessarily split by the projectivity of the $C_\bullet$.

That's why to build it, we first pick projective resolutions (A_i) of A and (C_i) of C, and add them together, giving us

$$\begin{array}{ccccccccccc} & 0 & & 0 & & 0 & & 0 & & \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & A_2 & \stackrel{a_2}{\to} & A_1 & \stackrel{a_1}{\to} & A_0 & \stackrel{a_0}{\to} & A & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & A_2 \oplus C_2 & & A_1 \oplus C_1 & & A_0 \oplus C_0 & & B & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & C_2 & \stackrel{c_2}{\to} & C_1 & \stackrel{c_1}{\to} & C_0 & \stackrel{c_0}{\to} & C & \to & 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & 0 & & 0 & & 0 & & 0 & & \end{array}$$

The middle line has projective elements, and some obvious maps $\begin{bmatrix} a_n & 0 \\ 0 & c_n \end{bmatrix}$, but doesn't yet have the map $A_0 \oplus C_0 \twoheadrightarrow B$. We can build that map from $A_0 \twoheadrightarrow A \hookrightarrow B$, and the projectivity of $C_0 \to C \twoheadleftarrow B$. But if we just bang this canonical and non-canonical choice together, the rightmost two squares won't commute.

We'll need to spoil the block diagonality of $\begin{bmatrix} a_n & 0 \\ 0 & c_n \end{bmatrix}$, with a map A_i to or from C_i. To build it, consider the sequences

$$\begin{array}{ccccccccc} \cdots & C_2 & \to & C_1 & \to & C_0 & \to & C & \to & 0 \\ & & & & & & & \| & & \\ \cdots & A_1 & \to & A_0 & \to & B & \to & C & \to & 0 \end{array}$$

We claim the second one is exact. (Since $A \xrightarrow{i} B$ is $1:1$, the kernel of $A_0 \to B$ is the same as that of $A_0 \to A$. Since $A_0 \to A$ is onto, the image of $A_0 \to B$ is the same as that of $A \to B$.) Now use proposition 5.1 to create vertical maps (t_i) making all squares commute.

Then if our middle sequence is defined as

$$A_n \oplus C_n \xrightarrow{\begin{bmatrix} a_n & (-1)^n t_n \\ 0 & c_n \end{bmatrix}} A_{n-1} \oplus C_{n-1}, \qquad A_0 \oplus C_0 \xrightarrow{\begin{bmatrix} i \circ a_0 & t_0 \end{bmatrix}} B$$

we get everything we want. (The $(-1)^n$ are because the t_i are constructed as degree 0 maps, but we use them as degree 1 maps.)

T **contravariant.** Everything works the same if we apply a contravariant functor T, except that our complexes are now backwards, which flips our definition of chain homotopies. To help keep track, we call the homology of the complex "cohomology", but the kernel mod image definition is the same. Now a left exact functor T like $\mathrm{Hom}(A, \bullet)$ gets right derived functors R^iT.

T **left exact.** Use an injective resolution instead.

7. $\mathrm{Ext}^\bullet$ GROUPS

7.1. **Using a projective resolution.** Define $\mathrm{Ext}^i(C, A) := (L_i\mathrm{Hom}(\bullet, A))(C)$, i.e. using a projective resolution of C and the contravariant right exact functor $\mathrm{Hom}(\bullet, A)$.

Stupid case: C is projective, so has a resolution $0 \to C_0 \to C \to 0$. Then we $\mathrm{Hom}(\bullet, A)$ the complex $0 \to C \to 0$ and find out $\mathrm{Ext}^{i>0}(C, A) = 0$.

Consider the category of SES $0 \to A \to B \to C \to 0$ with fixed A, C, and chain isomorphisms of SES. To such an SES, we look at the $\mathrm{Ext}^i(\bullet, A)$ functors, getting a LES

$$0 \to \mathrm{Hom}(C, A) \to \mathrm{Hom}(B, A) \to \mathrm{Hom}(A, A) \to \mathrm{Ext}^1(C, A) \to \cdots$$

Then 1_A maps to an element of $\mathrm{Ext}^1(C, A)$.

In particular, that element is 0 iff $\mathrm{Hom}(B, A) \to \mathrm{Hom}(A, A)$ has 1_A in the image, iff the SES is split. So we might hope that the element of $\mathrm{Ext}^1(C, A)$ measures the nonsplitness in some more general sense.

Theorem 7.1. *The map from isomorphism classes of extensions of C by A to $\mathrm{Ext}^1(C, A)$ is well-defined and bijective.*

Proof. First, well-defined. Given two isomorphic SES, apply $\mathrm{Ext}^\bullet$ as above. The third and fourth vertical maps come from the third and first in the SES, which are equalities, so we get a commuting square with vertical equalities. Hence the horizontal maps are the same.

For onto, let $g \in \mathrm{Ext}^1(C, A)$, and make a SES $0 \to K \overset{\kappa}{\hookrightarrow} F \twoheadrightarrow C \to 0$ where F is free. It gives the $\mathrm{Ext}(\bullet, A)$s

$$\cdots \to \mathrm{Hom}(K, A) \to \mathrm{Ext}^1(C, A) \to \mathrm{Ext}^1(F, A) = 0$$

i.e. g is the image of some $\bar{g} \in \mathrm{Hom}(K, A)$. Define

$$B := (A \oplus F) \Big/ \left(\begin{bmatrix} \bar{g} & \kappa \end{bmatrix} K \right)$$

Then we claim that $0 \to A \to B \to C \to 0$ is exact, and induces $g \in \mathrm{Ext}^1(C, A)$.

The best way to show injectivity is to say that we're defining different R-module structures $\cdot$ on the set $A \times C$, and figure out how to make those into an abelian group so that the map to $\mathrm{Ext}^1(C, A)$ is a group homomorphism. Then we already figured out that the kernel is 0. □

Example. There are two extensions of Z_p by Z_p: the trivial one and Z_{p^2}. Consider SES

$$0 \to Z_p \xrightarrow{pa\cdot} Z_{p^2} \xrightarrow{b\cdot} Z_p \to 0$$

These a, b live in $Z_p^\times$. By scaling by b^{-1}, we can reduce to $b = 1$. But then we still have $Z_p^\times$ different extensions (with the same group in the middle).

7.1.1. *Computing* Ext *groups of finitely generated* $\mathbb{Z}$*-modules.* If C or A is a direct sum, then the Hom groups and maps will also be direct sums, so the result will be a direct sum. The indecomposable options are $\mathbb{Z}$ and $\mathbb{Z}_{p^k}$, but we can handle $\mathbb{Z}_m$ just as easily.

- $C = \mathbb{Z}$. This was the stupid projective case from before. So $\mathrm{Hom}(C, A) \cong A$ and $\mathrm{Ext}^1(C, A) = 0$.
- $C = \mathbb{Z}_m$, with projective resolution $0 \to \mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z} \to 0$.
  - $A = \mathbb{Z}$. After Homming, the sequence is $0 \to \mathbb{Z} \xrightarrow{m\cdot} \mathbb{Z} \to 0$, with homology groups $\mathrm{Hom}(\mathbb{Z}_m, \mathbb{Z}) = 0, \mathrm{Ext}^1(\mathbb{Z}_m, \mathbb{Z}) = \mathbb{Z}_m$.
  - $A = \mathbb{Z}_n$. After Homming, the sequence is $0 \to \mathbb{Z}_n \xrightarrow{m\cdot} \mathbb{Z}_n \to 0$. Both Ext groups are $\mathbb{Z}_{\gcd(m,n)}$.

7.2. **Using an injective resolution.** Define $\mathrm{Ext}^i(A, B) := (R^i\mathrm{Hom}(A, \bullet))(B)$, from $\mathrm{Hom}(A, \bullet)$ being covariant left exact. To compute it, we start with an injective resolution

$$0 \to B \xrightarrow{b_0} B_0 \xrightarrow{b_1} B_1 \xrightarrow{b_2} B_2 \cdots$$

then

$$0 \to \mathrm{Hom}(A, B_0) \to \mathrm{Hom}(A, B_1) \to \mathrm{Hom}(A, B_2) \cdots$$

and look at

$$\mathrm{Ext}^1(A, B) := \{\phi : A \to B_1 \mid b_1 \circ \phi = 0\} \Big/ b_0 \circ \{\phi' : A \to B_0\}$$

...

Given a SES $0 \to A \hookrightarrow B \twoheadrightarrow C \to 0$, pick a function $\sigma : C \to B$ splitting $B \twoheadrightarrow C$. Then we get a symmetric map

$$(c_1, c_2) \mapsto \sigma(c_1 + c_2) - \sigma(c_1) - \sigma(c_2) \quad \in \ker(B \to C) = A$$

from $C \times C \to A$, making no reference to B.

...

7.3. **Universal coefficient theorems.** This is usually the first application of Ext groups.

Let $(C_\bullet, d^{-1})$ be a complex of **free** abelian groups to which we apply $\mathrm{Hom}(\bullet, A)$. Then we can compute the homology of the first, denoted $H_i(C)$, and the homology of the second, denoted $H^i(C; A)$ and called cohomology. We get a natural map

$$H^i(C;A) \to \mathrm{Hom}(H_i(C);A)$$

and the theorem is that (1) it fits into a canonical SES

$$0 \to \mathrm{Ext}^1(H_{n-1}(C), A) \to H^i(C;A) \to \mathrm{Hom}(H_i(C);A) \to 0$$

which (2) noncanonically splits (this uses the freeness).

The real reason that we only need Ext^1s is that this is about $\mathbb{Z}$-modules, which have very short resolutions; things are much more complicated over rings requiring longer projective resolutions.

This takes a while to prove (see e.g. `https://ncatlab.org/nlab/show/universal+coefficient+theorem`) so let's look at a slightly simpler case, computing homology of $C \otimes A$ in terms of that of C. Then the natural map

$$H_i(C) \otimes A \to H_i(C \otimes A)$$

(1) fits into the SES

$$0 \to H_i(C) \otimes A \to H_i(C \otimes A) \to \mathrm{Tor}(H_{i-1}(C), A) \to 0$$

where $\mathrm{Tor}(\bullet, A) = \mathrm{Tor}_1(\bullet, A)$ is the first derived functor of $\otimes A$, and (2) this sequence noncanonically splits.

7.3.1. *The proof.* Here's a trick: from C we consider the subcomplexes Z, B of kernels and images, on each of which the differential restricts (to 0). Then we get two SES of complexes:

$$0 \to B_\bullet \hookrightarrow Z_\bullet \twoheadrightarrow H_\bullet(C) \to 0 \qquad \text{all boundaries trivial}$$

and

$$0 \to Z_\bullet \to C_\bullet \to B_{\bullet-1} \to 0$$

where the latter splits row-wise (but not with commuting squares) *if each* C_i *is free.* Hence it still splits row-wise if we apply some covariant (say) functor T, so we again have a SES $0 \to T(Z_\bullet) \to T(C_\bullet) \to T(B_{\bullet-1}) \to 0$, inducing a LES

$$\cdots \to T(B_i) \to T(Z_i) \to H_i(TC) \to T(B_{i-1}) \to \cdots$$

which makes it look like it came from a SES $0 \to B \to Z \to C \to 0$, though it doesn't. (These shifts show up in the derived category picture.)

This breaks into SES

$$0 \to \mathrm{coker}\big(T(B_i) \to T(Z_i)\big) \to H_i(TC) \to \ker\big(T(B_{i-1}) \to T(Z_{i-1})\big) \to 0$$

To understand the first term, when T is right exact (e.g. $\bullet \otimes A$), apply it to the first SES obtaining an exact sequence

$$T(B_i) \to T(Z_i) \twoheadrightarrow T(H_i(C)) \to 0$$

letting us identify the first with $T(H_i(C))$.

For the third term, we want to get

$$L_1T(Z_i) \to L_1T(H_{i-1}(C)) \to T(B_{i-1}) \to T(Z_{i-1})$$

and have the first group vanish... so we obtain...

$$0 \to H_i(C) \otimes A \to H_i(C \otimes A) \to \mathrm{Tor}(H_{i-1}(C), A) \to 0$$

8. Interlude: filtrations on rings and modules

Let R be a ring. A **gradation** by an abelian group A is a direct sum decomposition $R = \oplus_{a \in A} R_a$ such that $R_a R_b \leq R_{a+b}$. If R is a vector space over an infinite field $\mathbb{F} \leq R_0$, then this is the same data as an action of the group $\mathrm{Hom}(A, \mathbb{F}^\times)$ on R by ring automorphisms. A **graded module** M is one with $M = \oplus_{a \in A} M_a$, where $R_a M_b \leq M_{a+b}$.

If A contains a submonoid A_+ we might be more interested in A_+-gradings, the most common case being $\mathbb{N}$-gradings. One nice thing about $\mathbb{N}$-graded rings R is that there is a map $R \twoheadrightarrow R_0$, not just $R_0 \hookrightarrow R$.

The naturals are ordered, and the reverse order is not isomorphic, allowing for two distinct notions: a **decreasing filtration** on a ring R is an $\mathbb{N}$-filtration $R = F_0 \subset F_1 \subset F_2 \subset \cdots$. One often adds the requirement that $\cap_i F_i = \{0\}$. Note that each F_i is an ideal, and one obvious way to make a decreasing filtration is to define $F_k = I^k$ for some ideal I; this is called the **I-adic filtration**.

From an $\mathbb{N}$-gradation one can construct a decreasing filtration, $F_i := \oplus_{k \geq i} R_k$. Conversely, from a decreasing filtration one can construct the **associated graded** ring $\oplus_i (F_i / F_{i+1})$. One composite is the identity, but the other usually is not.

All these definitions extend naturally to modules. If $R = R_0$ is just a field, then the associated graded of a module is unnaturally isomorphic to the original module.

Similarly we can define **increasing filtrations** $E_0 \subset E_1 \subset \ldots$ and perhaps ask that $\cup_i E_i = R$. In particular E_0 is a subring, and we can again speak of the associated graded. One reason to work with filtrations is that any sub or quotient ring of a filtered ring is again filtered (unlike the graded case). For example, TV is graded hence $U\mathfrak{g}$ is filtered. Then the PBW theorem is that $\mathrm{gr}\, U\mathfrak{g} \cong \mathrm{Sym}^\bullet \mathfrak{g}$.

Given an increasing filtration E on R, define the **Rees ring** $E[t]$ as the $R_0[t]$-subalgebra of $R[t]$ given by $\bigoplus_{i \in \mathbb{N}} R_i t^i$. Then for $\lambda \in R_0^\times$, $E[t]/\langle t - \lambda \rangle \cong R$, whereas $E[t]/\langle t \rangle \cong \mathrm{gr}\, R$. Geometrically, one should consider $R_0[t]$ as the ring of functions on a line, and its gradedness as an action of the multiplicative group. Then all the fibers are isomorphic except for the central one, which is different.

There is a version for decreasing filtrations as well. Let $F[t]$ be the $R[t]$-subalgebra of $R[t, t^{-1}]$ given by $\bigoplus_{i \in \mathbb{Z}} F_i t^{-i}$, where $F_{i \leq 0} := F_0 = R$. Once again, $F[t]/\langle t - \lambda \rangle$ is R for λ a unit, and $\mathrm{gr}\, R$ for $\lambda = 0$ (as long as one gives t degree -1). Finally, there's a version $\bigoplus_{i \in \mathbb{N}} F_i t^{-i}$ called the **blowup algebra** for its use in algebraic geometry (most often considered for I-adic filtrations).

9. Filtered complexes

Let $R = k[d]/\langle d^2 \rangle$, considered as a $\mathbb{Z}$-graded algebra with $\deg d = -1$. Then a graded module M is exactly a complex, and $\mathrm{ann}_d(M)/dM$ its homology.

What if, unrelated to this, M has a 2-step filtration $M \subset M^1$, where M^1 is a submodule (i.e. subcomplex)? (We'll keep the subscripts for M's being a complex.)

If M were actually graded ($M = M^0 \oplus M^1$), then we could compute its homology in each degree and add them together.

Question. What's the relation between $H(M)$ and $H(\mathrm{gr}\, M)$?

Obviously if $M = M_k$, then $H(M) = M_k$ but $H(\mathrm{gr}\, M) = \mathrm{gr}\, M_k$, so we can't compute $H(M)$ from $H(\mathrm{gr}\, M)$, only at best some sort of $\mathrm{gr}\, H(M)$.

Things can be much worse though. Example: let each M_i have basis t_i, dt_{i-1}, with M_i^1 spanned by just dt_{i-1}. Then M is exact, but M^1 and M/M^1 each have trivial differential and 1-D homology in every degree.

The basic picture, if we assume k is a field so that we can split M as $M^0 \oplus M^1$ where M^0 is *not* a subcomplex, is

$$\begin{array}{ccccccccc} \cdots & \to & M_i^0 & \to & M_{i-1}^0 & \to & M_{i-2}^0 & \to & \cdots \\ & & & \searrow & & \searrow & & & \\ \cdots & \to & M_i^1 & \to & M_{i-1}^1 & \to & M_{i-2}^1 & \to & \cdots \end{array}$$

where all two-step composites give 0.

The bottom row is a complex in itself, with homology $H(M^1)$. If we mod it out, we only see the top row, with homology $H(\mathrm{gr}\, M)$. What we've forgotten is the diagonal map.

Proposition 9.1. *There's a natural map* $d' : H^\bullet(M/M^1) \to H^{\bullet-1}(M^1)$*, and a filtration on* $H(M)$ *such that* $\mathrm{gr}\, H(M) \cong \ker d' \oplus \mathrm{coker}\, d'$.

Proof. Let $m + M_i^1 \in (M/M^1)_i$ be in $\ker d$. Then dm, a priori in M_{i-1}, is actually in M_{i-1}^1. Since $d(dm) = 0$, $dm \in \ker d$. Then it remains to check that if we change m by adding dm', it doesn't change dm.

The filtration on $H(M)$ is derived from the filtration on M. It is then tiresome to correspond the two spaces. □

Question. What if M has a longer filtration?

Say, $M \supseteq M^1 \supset M^2$. Now in addition to

$$\begin{array}{ccccccccc} \cdots & \to & M_i^0 & \to & M_{i-1}^0 & \to & M_{i-2}^0 & \to & \cdots \\ & & & \searrow & & \searrow & & & \\ \cdots & \to & M_i^1 & \to & M_{i-1}^1 & \to & M_{i-2}^1 & \to & \cdots \\ & & & \searrow & & \searrow & & & \\ \cdots & \to & M_i^2 & \to & M_{i-1}^2 & \to & M_{i-2}^2 & \to & \cdots \end{array}$$

we'd have knight's-moves maps to the SSE. One can again define the Southeast map d', but now it is only a differential on $H(\mathrm{gr}\, M)$, so instead of $\ker \oplus \mathrm{coker}$ one should take *its* homology, $H(H(\mathrm{gr}\, M), d')$. But we're not done: the SSE maps give us *another* differential d'' on there. Then the statement is that

$$H(H(H(\mathrm{gr}\, M), d'), d'') \cong \mathrm{gr}\, H(M)$$

for some filtration on $H(M)$.

You can now imagine a mathematical object, called a "page of a spectral sequence", where we have a grid of modules and a collection of semi-Southeast differentials, whose homology gives the "next page".

9.1. **Double complexes.** Consider a commuting grid of modules, where every row and column is a complex, and call this a **double complex**. Then we can $\oplus$ along the diagonals (which will be finite sums if this grid is supported in the correct quadrant, in case we care), but to make the result a complex, we have to introduce signs on every other row.

Now, however, this resulting single complex comes with filtrations, by either the rows or the columns of the original double complex.

One natural way to obtain such a double complex is to start with a single complex, and resolve every element (say, projectively).

Proposition 9.2. *Let* $M_{\bullet\geq 0,\bullet\geq 0}$ *be a double complex whose vertical homology is supported in the* 0*th row, i.e. is the complex* $(H_{\bullet\geq 0,0})$*. Consider* H *as a (rather trivial) double complex too, so the map* $M \to H$ *gives an isomorphism of vertical cohomology. Then the flattening of* M *also has a map to* H*, now inducing an isomorphism of horizontal cohomology.*

10. The Grothendieck spectral sequence

Let M be a module and S, T two right exact covariant functors (there are many other versions of course). Then $S \circ T$ is also right exact, and we should be able to compute $L_\bullet(S \circ T)(M)$.

Start with a projective resolution $\cdots \to M_1 \to M_0 \to 0$ whose only homology is $H_0 = M$. Apply T to obtain a complex. What we want to do now is apply S as well, then take the homology, giving $L_\bullet(S \circ T)(M)$.

But instead we resolve the complex $T(M_\bullet)$ to a double complex, whose flattening would give the same homology, namely $(L_\bullet T)(M)$. *Now* apply S, obtaining a double complex whose flattening would give the thing we want $(L_\bullet S \circ T)(M)$.

The benefit of having done this in two steps is that the double complex comes filtered. So if we take vertical cohomology obtaining $(L_\bullet S)(L_\bullet T)(M)$ first, we can use the filtered story from before to continue taking homologies, in the limit obtaining a gr of $L_\bullet(S \circ T)(M)$.