Math 6520 final exam, due December 17 2019

You may draw upon your own notes, homeworks, and [Guillemin-Pollack], but not other sources. In particular, don't turn to the back to get started until you've finished studying what you want to from other sources.

1a. Let C be an open cylinder, $S^1 \times \mathbb{R}$. State the Künneth formula and use it to compute the cohomology ring of C.
1b. State the Mayer-Vietoris formula for compactly supported cohomology and use it, too, to compute the compactly supported cohomology of C (again the open cylinder).
2. Let Mö be the open Möbius strip. Use the Mayer-Vietoris formula for compactly supported cohomology to compute the compactly supported cohomology of Mö.
3. Let $V \to M$ be an oriented 2-dimensional vector bundle. Show that the compactly supported Euler characteristic (alternating sum of dimensions of compactly supported cohomology) of $V \setminus M$, i.e. with the zero section ripped out, is zero.
4. Define the Lefschetz number by the supertrace of the action on cohomology (so, in particular, it is defined even when M is noncompact). Let $\tau : M \to M$ be a self-map preserving each of the pieces in an open cover $M = U \cup V$. Show how to compute $L_\tau(M)$ from L_τ of $U, V, U \cap V$.
5a. Let $E = \{\vec{v} \in \mathbb{R}^2 \ : \ |\vec{v}| \geq 1\}$. Put an atlas on E to show that it is a manifold-with-boundary.
5b. Let $F = E/\sim$, where the equivalence classes are singleton except for $\vec{v} \sim -\vec{v}$ when $|\vec{v}| = 1$. Put an atlas on F making it a manifold, in such a way that the natural map $E \to F$ is smooth.
6. Let $A, B \subseteq M$ all be compact oriented, with $\dim A + \dim B = \dim M$, but don't assume A, B transverse. Rather assume that for all $c \in A \cap B$, the dimension $\dim(T_c A + T_c B)$ is constant. Let $V = \{(c \in C, \vec{v} \in T_c M) \ : \ \vec{v} \perp T_c A + T_c B\}$. Show that $I_M(A, B) = I_V(C, C)$.
9. Let M be oriented. Let $C_i(M)$ be the space of (finite) formal linear combinations of maps $\phi_N : N \to M$, where N varies over compact oriented $(\dim M - i)$-manifolds-with-boundary. Recall this forms a complex with boundary map $\partial : C_i(M) \to C_{i+1}(M)$. Call its cohomology $\mathbb{B}$.
Define the map $C_i(M) \to \Omega^{\dim M - i}(M)^*$, $\phi_N \mapsto \left(\alpha \mapsto \int_N \phi_N^*(\alpha)\right)$. Show that this descends to a map on the cohomologies of the two complexes.
Show that this gives a map $\mathbb{B}_i \to H_c^i(M)$. Is it an isomorphism?